Political Competition with Endogenous Party Formation and Citizen Activists

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Abstract

The paper studies political competition between endogenously formed parties. For this purpose, it develops a theoretical model in which party formation allows like-minded citizens to coordinate their political behavior in two ways. First, they can share the cost of running in a public election. Second, they can select a candidate for this election from their ranks, thereby committing to a policy platform. The paper characterizes the set of political equilibria with two competing parties and with one uncontested party. In particular, it studies the policy platforms that can be offered by stable political parties. In equilibria with two competing parties, the distance between both platforms is always positive but limited, in contrast to both the median voter model and the citizen candidate model. In equilibria with one uncontested party, the median voter can be worse off than in equilibria with two competing parties.

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1 Introduction

This paper studies electoral competition between political parties that are endogenously formed by policy-interested citizens within the political process. The analysis explicitly accounts for two central aspects of the empirically evident group-character of political parties: First, the foundation and the composition of parties result from the strategic interaction of heterogeneous citizens. Second, the nomination of party candidates and the selection of policy platforms result from the strategic interaction of heterogeneous party members. The paper thereby contributes to the economic theory of electoral competition, initiated by Downs (1957). Most previous papers in this field have studied elections between independent candidates or parties that act as unitary agents. In particular, there is no party formation in the commonly used workhorse models of political economy, including the classical median voter model (Downs 1957) and the citizen candidate model (Osborne & Slivinski 1996, Besley & Coate 1997). The main goal of this paper is to investigate the effects of endogenous party formation on the policies implemented in political equilibria. More precisely, I study which policy platforms can be offered by political parties with stable membership structures, i.e., with sets of party members such that no citizen has an incentive to change his party affiliation.

For this purpose, I develop a formal model of political competition with endogenous party formation. This model has four central features. First, the political process is democratic in every respect: there is a (large) set of citizens with heterogeneous policy preferences each of whom is entitled to join a party, to become the party's candidate for a public office and to vote in a public election for this office. Both the number and the composition of the competing parties are determined endogenously within the political process. Second, political activity is costly: Parties have to pay an exogenous cost of running to enter political competition, and citizens have to pay an exogenous membership cost to join a party. The members' payments are used to finance the cost of running mentioned above. Third, party members coordinate their behavior in primary elections. The members of each party select one candidate from their ranks to run for the public office. This allows the party members to commit to the candidate's ideal policy, thereby choosing the party's policy platform. Fourth, there is electoral risk. In particular, the citizens perceive the median voter's ideal policy as the realization of a random variable with a probability distribution that satisfies a set of regularity conditions.

The model builds strongly on the citizen candidate approach by Osborne & Slivinski (1996) and Besley & Coate (1997). In particular, the two first features are similar in spirit to the citizen candidate model, but are adapted to suit a model

with endogenous party formation. In contrast, the modeling of endogenous party formation and within-party coordination (third feature) and the assumption of electoral risk (fourth feature) differ from the previous literature. The latter modeling decision helps to make crucial trade-offs and mechanisms at work more visible and less degenerate. The exogenous parameters of the model are given by the citizens' cost of party membership, the parties' cost of running and the degree of electoral risk implied by the median voter distribution. The paper derives comparative statics results for these parameters as well as limit results for the cases of full electoral certainty and zero membership costs.

The paper provides three contributions to the theoretical literature on electoral competition. First, I develop a novel theoretical framework that allows to study party formation and political competition simultaneously. From a conceptual perspective, the identification of policy platforms that can be supported by stable party membership structures is novel to the literature on electoral competition. In particular, a party can be stable if and only if (i) no party member can benefit from leaving the party and saving the membership cost, and (ii) no independent agent can benefit from joining a party and potentially becoming its candidate. Otherwise, some citizen has an interest to change his party affiliation, anticipating the effect on the parties' policy platforms and the implemented policy. From a theoretical perspective, the second condition is particularly interesting because it requires to jointly analyze the incentives of independent agents to join a party and their prospects of being selected as candidates by the other party members. As main steps of the analysis, I show that the citizens' implied preferences over both party membership and the parties' policy platforms satisfy versions of the single-crossing condition by Gans & Smart (1996).

The second contribution is given by a complete characterization of the policy platforms that can be offered in political equilibria with two competing parties and with one uncontested party. For the main result of this paper, I revisit the classical question whether the equilibrium platforms of two competing parties are fully convergent (as in Downs 1957) or strongly divergent (as in Osborne & Slivinski 1996 and Besley & Coate 1997). In contrast to the results of these previous papers, I show that the platform distance is always strictly positive but limited. Intuitively, endogenous party formation gives rise to both a centrifugal and a centripetal force. On the one hand, parties can only be stable if their policy platforms are sufficiently different - otherwise, no citizen would be willing to support a party and bear the membership costs. On the other hand, parties can only be stable if their policy platforms are sufficiently close to each other - otherwise, moderate independent

citizens would benefit from joining a party and becoming its candidate, thereby reducing the platform distance. In the benchmark case of full electoral certainty, the policy platforms in two-party equilibria are uniquely determined and exhibit a strictly positive distance that depends only on the cost of party membership. This limit case hence makes the difference to the results of both the median voter model and the citizen candidate model most obvious. In particular, endogenous party formation substantially reduces the multiplicity of equilibria in the citizen candidate model, where the equilibrium platforms may even be extremely distant.[1]

Finally, I derive a novel result on the difference between one-party and two-party equilibria, which sheds light on the desirability of (multi-party) democratic competition. In particular, I show that the platform of an uncontested party can deviate more strongly from the median voter position than the platforms of two competing parties, as long as the electoral risk is sufficiently small. Hence, the median voter is ex post worse off in some one-party equilibria than in every two-party equilibrium. Importantly, the citizen candidate models by Osborne & Slivinski (1996) and Besley & Coate (1997) give rise to the opposite conclusion.[2] The result of this paper is however in line with the conventional view that democratic competition (between multiple parties) is beneficial because it provides incentives for each party to respect the voters' interests. Intuitively, the members of competing parties are forced to compromise between their own policy preferences and those of the electorate at large, while the members of an uncontested party can focus on the preferences within their own group.

The paper proceeds as follows. Section 2 discusses the related literature, and Section 3 presents the model. Sections 4 to 6 solve the electoral game and derive the set of political equilibria with two competing parties and with one uncontested party for a fixed combination of the exogenous parameters. Section 7 provides comparative statics with respect to the membership costs and the degree of electoral risk. Additionally, it contains results for the limit cases of electoral certainty and zero cost of party membership. Section 8 concludes. All formal proofs are relegated in the appendix.

[1] As acknowledged by Besley & Coate (1997), the multiplicity of equilibrium platforms represents a dissatisfactory feature of the citizen candidate model, ruling out clean empirical predictions. See also Dhillon & Lockwood (2002), De Sinopoli & Turrini (2002) and Roemer (2003).

[2] In both models, the median voter is strictly better off in each equilibrium with one uncontested candidate than in any equilibrium with two competing candidates.

2 Related literature

The model builds on the citizen candidate framework introduced by Besley & Coate (1997) and Osborne & Slivinski (1996). In both versions of this model, the set of candidates is determined endogenously from the set of citizens who are not only entitled to vote in a democratic election, but can also decide to run as (individual) candidates, facing an exogenous cost of candidacy. There are no parties, and citizens cannot coordinate their political behavior. The models do not deliver a unique theoretical prediction but a multiplicity of political equilibria with either one or two candidates. Their main insight is that the endogeneity of the candidate set eliminates the possibility of completely convergent platforms in two-candidate equilibria. This impossibility result is in sharp contrast to the classical predictions of the median voter model by Downs (1957) and the probabilistic voting model by Lindbeck & Weibull (1987), but is in line with empirical observations. In both versions of the citizen candidate model, there may however be equilibria with arbitrarily polarized candidates. In the model by Besley & Coate (1997), the platform distance in two-candidate equilibria is only bound by the extremes of the policy space.[3]

A number of papers extend the basic citizen candidate framework to accommodate political parties. For example, Rivière (1999) studies the formation of parties as cost-sharing devices and provides a game-theoretical explanation for Duverger's law, i.e., the prevalence of two-party systems under the plurality rule. The same result is derived in a different environment by Osborne & Tourky (2008), who analyze the incentives to form parties within a group of legislators under the assumptions of costly participation and economies of party size. In contrast, Levy (2004) examines whether the formation of political parties can be effective in the sense that it enables offering platforms that would not be feasible without parties. Morelli (2004) studies the implications of alternative electoral systems for the formation of parties by agents with heterogeneous policy preferences. Snyder & Ting (2002), as well as Poutvaara & Takalo (2007), show that parties may serve as brand names or screening devices, which provide superior information about the candidates' preferences or quality, respectively.

In contrast to this paper, these papers do not examine the effects of endogenous formation of political parties on political polarization. Directly related to this issue, they do not show that party formation alleviates the indeterminacy of the basic

[3]In the version of Osborne & Slivinski (1996), there is large set of equilibria with potentially large, but limited polarization. In contrast to the analysis in this paper, however, the upper bound on the platform distance results from the assumption of sincere instead of strategic voting and is not related to the candidates' behavior or coordination.

citizen candidate model. Furthermore, these papers either consider only the case of electoral uncertainty or strongly restrict the type space. In this paper, I instead study the implications of endogenous party formation on platform choice in a general setting, allowing for different degrees of electoral uncertainty as well as a continuum of agents without restrictions on the location of bliss points.[4]

To my knowledge, only one previous paper investigates the effect of political parties on platform choice within the citizen candidate framework. Cadigan & Janeba (2002) study party competition in a US-style presidential election with primary elections and identify a strong connection between membership structures and party platforms. Instead of endogenizing membership decisions, however, they assume that all citizens have exogenous party affiliations. The drawback of this modeling is that any combination of platforms represents a political equilibrium for some membership structures. As they cannot distinguish between stable and unstable membership structures, the model only delivers limited insights into the effects of party formation.[5]

In addition, there is a small number of papers on the formation of political parties outside the citizen candidate framework. Most closely related, Roemer (2006) studies the effects of endogenous party formation and campaign contributions by policy-motivated citizens. Similar to my model, the unique political equilibrium of Roemer's model features positive but limited platform distance. However, both models differ considerably in many aspects. Most importantly, Roemer applies the cooperative notion of "Kantian equilibrium" in which agents consider joint (proportional) deviations of all party members at the contribution stage. The implications of this equilibrium concept differ strongly from the non-cooperative notion of Nash equilibrium applied below.[6] Furthermore, the platforms are chosen through a Nash bargaining process in which the agents' influence is proportional to their individual contributions in his model. In my model, in contrast, there are primary elections in which each party member has exactly one vote.

In other papers, citizens only decide whether to support exogenously given political parties by contributing to their electoral campaigns (Herrera et al. 2008, Campante 2011, Ortuño-Ortin & Schultz 2005). Although there is no endogenous party formation in these models, citizens have an indirect influence on the policy platforms that are chosen by the parties, taking into account the induced contribution

[4]Dhillon (2004) surveys the existing theoretical models with pre-election as well as post-election party formation, with a particular focus on papers that extend the citizen candidate model.

[5]Furthermore, Cadigan & Janeba (2002) do not allow for electoral uncertainty.

[6]For example, every citizen is member of one party in the model of Roemer (2006) while there is a (large) set of independents in any equilibrium of my model.

behavior. Poutvaara (2003) also studies a model with endogenous party formation, which predicts a positive but limited platform distance. However, the results are mainly driven by the assumption that agents make their membership decisions based on expressive objectives while, in my model, they follow from strategic membership decision and cooperation between like-minded citizens.[7]

Finally, this paper also relates to the literature on probabilistic voting and electoral uncertainty, beginning with the seminal paper of Lindbeck & Weibull (1987). Eguia (2007) studies the effect of electoral uncertainty in the citizen candidate model. Without party formation, electoral uncertainty has the effect of increasing the set of political equilibria with two candidates by allowing for asymmetric equilibria. However, electoral uncertainty per se does not lead to additional centripetal forces and does not limit political polarization. Both models focus on the behavior of individual agents and do not examine the effects of party formation.

3 The model

I start by specifying the basic setting of the model, including the set of agents, their preferences and the policy space. Subsequently, I explain the political process and define the notion of a political equilibrium.

3.1 The environment

The set of agents I is given by a continuum of citizens of mass one, with typical element i. The utility of each citizen i depends on some implemented policy $x \in X$ and on two payments $\alpha_i^L \in [0, \infty)$ and $\alpha_i^R \in [0, \infty)$ he makes. As will become clear below, these payments can be interpreted as party contributions. The policy space X is given by the real line $(-\infty, +\infty)$. Each agent i has linear Euclidean policy preferences with a unique ideal point $w_i \in X$. Formally, the preferences of citizen i can be captured by the utility function

$$u\left(x, \alpha_i^L, \alpha_i^R; w_i\right) = \underbrace{-\left|x - w_i\right|}_{=:v_i(x)} - \alpha_i^L - \alpha_i^R \,. \tag{1}$$

In the following, I refer to $v_i(x) = -\left|x - w_i\right|$ as i's policy payoff.

[7] Besides, there exist a few models on endogenous formation of political parties under proportional electoral systems in which the implemented policy is given as a weighted sum of the party platforms (e.g. Gomberg et al. 2004, Gerber & Ortuño-Ortin 1998). Due to the incentives that stem from this electoral system, these models typically predict an extremely high level of political polarization.

The agents have heterogeneous ideal points, and the distribution of ideal points over I has full support on X. For most of this paper, I assume that this distribution is not known ex ante. In particular, the median voter's bliss point is commonly perceived to be the realization of a random variable with *cumulative distribution function* Φ and corresponding *probability density function* ϕ.[8] I assume that Φ satisfies the following regularity conditions.

Assumption 1. *The distribution function* Φ *is twice continuously differentiable, log-concave and symmetric with* $\Phi(x) = 1 - \Phi(-x) > 0$ *for all* $x \in \mathbb{R}$. *Moreover, it satisfies* $\lim_{x \to -\infty} \Phi(x) = 0$.

Assumption 1 is satisfied by many commonly used distribution functions, including the normal, the logistic and the Laplace distributions.[9] Implicitly, it implies that the expected value of the population median is normalized to 0, and that the density ϕ has full support on $\mathbb{R}$.

3.2 The political process

The implementation of policy x follows from a political process that involves four stages: the party formation stage, the candidate selection stage, the general election stage and the policy implementation stage. The entire process is structured by two parties, denoted by L and R.[10]

At the first stage (party formation), each citizen i chooses his payments $\alpha_i^L \in [0, \infty)$ and $\alpha_i^R \in [0, \infty)$. He becomes a *member* of party $Q \in \{L, R\}$ if α_i^Q exceeds some exogenous number > 0. The member set of party Q is denoted by $M^Q = \left\{ i \in I : \alpha_i^Q \geq c \right\}$. I assume that each citizen can be a member of at most one party.[11] The membership cost c can be thought of as a monetary payment, but also as hours worked and effort spent for the party's electoral campaign and party meetings. Party Q becomes *active* if the sum of its contributions $\sum_{i \in I} \alpha_i^Q$ is weakly above the threshold C, which can be interpreted as an exogenous cost of running in the general election (as in Besley & Coate 1997 and Osborne & Slivinski 1996). To

[8] This assumption introduces electoral risk: ex ante, the outcome of an election between two parties with different policy platforms is uncertain.

[9] More specifically, log-concavity is a standard regularity condition that implies a monotonically decreasing hazard rate $\Phi(x)/\phi(x)$. The assumption of symmetry is in line with most previous papers on electoral competition with electoral uncertainty. The last sentence rules out cases in which electoral results do not depend on the parties' policy platforms at all.

[10] The restriction to only two parties is made for the sake of concreteness and simplicity. In the following, I provide results on equilibria in which either one or two parties become *active*. All results remain valid if there are more than two potentially active parties.

[11] This is without loss of generality: In equilibrium, no citizen ever wants to be a member of both parties.

rule out the prevalence of degenerate one-member parties in equilibrium, I assume that C is strictly larger than $2c$.

At the second stage (candidate selection), the members of each *active* party $Q \in \{L, R\}$ select one agent from their ranks as candidate for the following general election. In particular, a series of pairwise elections between all agents in the member set M^Q is conducted. The party candidate is chosen with equal probability from the subset of members that do not lose against any other agent in M^Q, i.e., from the set of Condorcet winners. As will become clear below, the ideal point of the selected candidate can also be interpreted as the party's policy platform $q \in \{l, r\}$. I assume that, at this stage, the members of party Q are informed about the median voter distribution Φ and the bliss points of all members of their own party, but not about the bliss points of the other party's members. Hence, they select a candidate from their ranks, holding some belief $-\hat{q}$ about the competing party's platform.[12] Figure 1 in Appendix B illustrates this information structure.

At the third stage (general election), the candidates of all *active* parties run in a general election and their ideal points become publicly observable. The candidates are unable to make binding policy commitments. Each citizen $i \in I$ casts his vote for one of these candidates. If there are two active parties, the candidate with the higher share of votes becomes the president. If both candidates receive the same share of votes, the president is determined by tossing a fair coin. If there is only one active party, the candidate of this party directly becomes president. If there is no active party, the presidential position remains empty.[13]

At the final stage (policy implementation), the president independently chooses the implemented policy $x \in X$. If there is no president, no policy is implemented and the utility of all citizens equals $-\infty$.

3.3 Allocations and political equilibria

A strategy β_i of agent i specifies his contributions (α_i^L, α_i^R), his voting behavior at the primary election stage for all possible member sets (M^L, M^R), his voting behavior at the general election stage for each combination of policy platforms (l, r) and his policy choice in case of becoming president. An allocation is given by a partition of the population into the sets of party members $\left(M^L, M^R\right)$ and the set

[12] This information structure simplifies the following equilibrium analysis, because changes in the member set M^L may affect the choice of platform l, but not the choice of platform r. Qualitatively, all results of this paper would however be identical if all party members could observe M^L as well as M^R at the candidate selection stage.

[13] Note that the vote result reveals whether the median voter's ideal point is closer to l or r. At the previous stages of the political process, the agents are not able to draw inferences on the median voter position from the member sets or the policy platforms.

of independent agents $I \setminus \left(M^L \cup M^R\right)$, and a tuple of party platforms (l, r). With some abuse of notation, I denote the platform of an inactive party Q by $q = \emptyset$.

A political equilibrium is given by a Perfect Bayesian equilibrium of the game defined above, consisting of a strategy profile β and a belief system such that, first, the strategies of all agents are sequentially rational – maximizing the agents' individual utilities – given the belief system and, second, the belief system is consistent with the strategy profile β everywhere on the equilibrium path. Additionally, I assume that the agents' equilibrium strategies do not involve weakly dominated actions at the candidate selection stage, and that the agents vote sincerely at the general election stage.[14]

In the next sections, I investigate the set of allocations in Perfect Bayesian equilibria, i.e., the set of stable member sets (M^L, M^R) and corresponding policy platforms (l, r). I concentrate on equilibria in pure strategies.[15] I solve the model backwards starting with the policy implementation stage.

4 Policy implementation and general election

The last two stages of the game can be solved straightforwardly. At the final stage, the elected president chooses policy $x \in X$ to maximize his individual policy payoff. If agent j is the president, his optimal action is to choose x equal to his own ideal point w_j. This policy choice is anticipated by all agents at the previous stages. Thus, the nomination of agent i as presidential candidate by party Q implies a credible commitment to the policy $q = w_j$. In the following, I hence refer to policy l (r) as the policy platform of party L (R).

At the general election stage, all citizens observe the platforms of all active parties and cast their votes for one of them. Assume that both parties are active. Anticipating the policy choices of both presidential candidates, citizen i sincerely votes for the party whose platform is closer to his own ideal point w_i.

As individual utilities are single-peaked in x, the median voter's preferred party always receives a strictly larger share of votes. As a convention, let the platform of party L be located left of the platform of party R, $l \leq r$. Then, party L (R) wins the election if the median voter's ideal point m is located to the left (right) of $(l + r)/2$.

[14] With a finite set of voters and two alternatives, sincere voting would be the only weakly dominant strategy. With a continuum of voters, the notion of weak dominance is not properly defined since no voter can ever be pivotal. Note that the member set of each active party turns out to be finite in any political equilibrium.

[15] In general, there may also exist political equilibria in mixed strategies.

Ex ante, all citizens only know the probability distribution Φ of the median voter's ideal point $m \in \mathbb{R}$. Hence, the commonly perceived winning probability $p(l, r)$ of party L is given by

$$p(l, r) = \Phi\left(\frac{l + r}{2}\right) \in (0, 1)\,. \tag{2}$$

Under Assumption 1, there is electoral risk: ex ante, all citizens assign strictly positive winning probabilities to both parties for any tuple (l, r). Note also that the winning probability $p(l, r)$ is continuously increasing both in l and r. For party members, the choice between alternative policy platforms hence involves a smooth trade-off between electoral prospects and the subjective desirability in case of winning.

5 Candidate selection and platform choice

At the candidate selection stage, the members of each *active* party Q select a candidate from their ranks, taking into account his perceived electoral prospects. To simplify the exposition, the following sections focuses on candidate selection in the leftist party L. Assume that party L has received sufficient contributions to become *active*. To avoid case distinctions, I concentrate on allocations in which each party has an odd number of members.[16]

At this stage, the member sets M^L and M^R have been determined as the outcome of the party formation subgame at the first stage. The members of party L select a candidate and thereby platform l through a series of pairwise elections between all agents in M^L. They are able to observe the ideal points of all members in L and, in particular, to identify the median party member.[17] For simplicity, I denote by m^L the median member's ideal policy in the following. Crucially, voting behavior in these pairwise elections depends on the expected platform $\hat{r}$ of the competing party R.[18] Moreover, the members of party L take into account the winning probability (2) as implied by the expected platform $\hat{r}$ and the median voter distribution Φ. I distinguish between two cases in the following.

First, let the members of L expect party R to remain inactive. With some abuse of notation, I denote their platform belief by $\hat{r} = \emptyset$ in this case. The candidate

[16] Allowing for allocations with an even number of party members slightly complicates the exposition, but has no effects on the qualitative results of this paper.

[17] Formally, I define the median party member as the member j such that equally many agents in M^L have a weakly smaller ideal point and a weakly larger ideal point than w_j.

[18] Recall that the members of L are unable to observe the ideal points of party R's members.

selection procedure gives rise to a straightforward outcome.

Lemma 1. *For platform belief $\hat{r} = \emptyset$, the candidate of party L is given by the median party member with ideal point m^L.*

If party L is the only active party, its candidate will become president and implement his ideal policy l with certainty. Hence, the members of L effectively choose the implemented policy x when voting on platform l. Their preferences over the platform l coincide with their single-peaked policy preferences $v_i(x)$. Therefore, a standard median voter result is result is reproduced: With single-peaked preferences, the median voter's preferred option is the unique Condorcet winner and will hence win a pairwise election against any other option. At the candidate selection stage, the median voter is given by the median party member and his preferred option is given by his own ideal point m_L.

Second and more interestingly, let the members of L expect party R to become active and its platform to be $\hat{r} \geq m_L$. By Lemma 2, pairwise elections again give give rise to a clear-cut decision.

Lemma 2. *Let $\lambda(\hat{r}, M^L) \coloneqq \arg \max_{\{w_i:\ i \in M^L\}} p(w_i, \hat{r})(\hat{r} - w_i)$. For any platform belief $\hat{r} \geq m^L$, the candidate of party L is given by a member with ideal point equal to $\max\left\{m^L, \lambda(\hat{r}, M^L)\right\}$.*

Lemma 2 follows from two insights that generalize beyond the details of this model. For the first insight, note that the members of L use the candidate selection subgame to choose their preferred platform in a public election against policy $\hat{r}$. The member's implied platform preferences are in general not single-peaked. But, as I show in the formal proof of Lemma 2, these preferences generically satisfy the single-crossing property by Gans & Smart (1996) for any distribution function Φ. Consequently, voting is monotonic in any pairwise election. Again, the median member's preferred platform represents a Condorcet winner in any set of available platforms and given any composition of the membership set M^L. Pairwise elections are one among many voting protocols for which a Condorcet winner prevails whenever it exists.[19]

For the second insight, consider a party member i with some bliss point $\omega_i < \hat{r}$. Conditional on platform $\hat{r}$ and belief $\hat{r}$, his expected policy payoff is given by

$$\tilde{v}_i(l, \hat{r}) \coloneqq p(l, \hat{r})(-|l - w_i|) + [1 - p(l, \hat{r})]\,(-|\hat{r} - w_i|) \tag{3}$$

[19]For example, primary election would lead to the same platform choice if the median party member would be entitled to nominate his preferred member as in Poutvaara (2003). Also, if all party members were entitled to vote and to run as candidates, the unchallenged candidacy of the Condorcet winner identified above would represent a subgame equilibrium.

For agent i, his own ideal point w_i strictly dominates all platforms to the right of $\hat{r}$ as well as to the left of w_i.[20] In the following, we can thus focus on the remaining interval $[\omega_i, \hat{r})$.

For platforms in the interval $[\omega_i, \hat{r})$, i's expected payoff simplifies to

$$\tilde{v}_i(l, \hat{r}) = p(l, \hat{r})(\hat{r} - l) + w_i - \hat{r} \ . \tag{4}$$

In this interval, i's platform preferences hence involve an intuitive trade-off between the probability of winning and the subjective desirability in case of winning: When platform l is raised towards $\hat{r}$, agent i on the one hand benefits from an increasing winning probability $p(l, r)$ of party L. On the other hand, an increase in l lowers his utility conditional on L's victory, $-|l - w_i|$. Among all available platforms in $[w_i, \hat{r}]$, i prefers the one that maximizes the auxiliary function

$$\Gamma(l, \hat{r}) \coloneqq p(l, \hat{r})(\hat{r} - l) \ . \tag{5}$$

For any $l < \hat{r}$, the activity of party L leads to a left-shift of the *expected policy* $p(l, \hat{r})l + [1 - p(l, \hat{r})]\,\hat{r}$. Function Γ measures the size of this effect, and is henceforth referred to as the *policy effect function*. As I show in the appendix, Γ is strictly quasi-concave and has a unique maximizer in $(-\infty, \hat{r})$ under Assumption 1.[21]

Lemma 2 follows from the combination of these two insights: The members of party L always choose the platform that maximizes $\Gamma(l, \hat{r})$ over the set of available platforms in the interval $\left[m^L, \hat{r}\right)$. In this model, the set of available platforms is given by set of bliss points of party L's members. Hence, platform l is either given by the constrained maximizer $\lambda(\hat{r}, M^L)$ or by the ideal point m^L of the party median m^L, whatever is larger.

6 Political equilibria

In the previous sections, I have investigated the outcome of the candidate selection stage in party L for any combination of member set M^L and platform belief $\hat{r}$. In a political equilibrium, this platform belief must be consistent: The selected candidate

[20] Intuitively, this strict dominance can be explained as follows. First, platform $l = w_i$ shifts the expected policy $p(l, \hat{r})l + [1 - p(l, \hat{r})]\,\hat{r}$ from $\hat{r}$ towards w_i, while all platforms $l \geq \hat{r}$ shift the expected policy even further away from w_i. Second, platform w_i comes with a larger probability to win against party R as well as a larger policy payoff in case of winning than every platform $l < w_i$, while the payoff in case of losing against R stays constant.

[21] In particular, Γ is strictly quasi-concave whenever Φ is log-concave, i.e., has a monotonically increasing hazard rate. Figure 2 in Appendix B depicts the policy effect function graphically for Φ given by a normal distribution.

with ideal point l must prevail in the pairwise elections of party L given the correct belief $\hat{r} = r$. If party R is active, its platform r must satisfy a corresponding condition. If the membership structures were given exogenously by some partition (M^L, M^R), then these conditions would already pin down the unique equilibrium combination of policy platforms.

In the game studied here, however, the citizens choose their party affiliation endogenously at the first stage, anticipating the effects of their choice on the platforms (l, r) and ultimately on the implemented policy x. In a political equilibrium, membership structures must therefore be stable in the sense that

(I) no member of a party $Q \in \{L, R\}$ can profitably leave his party,

(II) no independent citizen can profitably join a party $Q \in \{L, R\}$,

(III) no member of a party $Q \in \{L, R\}$ can profitably change his party affiliation.

Conditions (I) to (III) are necessary and jointly sufficient conditions for a political equilibrium. In the following, I use (I) and (II) to derive the set of platforms pairs (l, r) that can be supported by some stable membership structure (M^L, M^R). It turns out that condition (III) does not restrict the set of equilibrium platforms further.

Henceforth, I refer to an allocation as *exit-stable* if it satisfies (I), and as *entry-stable* if it satisfies (II). I start by investigating the implications of *exit-* and *entry-stability* for the set of equilibria in which both parties L and R are *active* in Subsection 6.1. In the following Subsection 6.2, I proceed with the set of equilibria with only one *active* party.

6.1 Political equilibria with two active parties

I start by deriving three necessary properties of equilibrium allocations with two active parties. For the first property, consider an allocation with two *active* parties, i.e., in which the sum of contributions to each party $Q \in \{L, R\}$ is larger than the exogenous cost of running C. In the following, party Q is said to be *efficient* if $\sum_{i \in I} \alpha_i^Q \in [C, C+c)$ holds, i.e., if there is no wasteful over-contribution and the withdrawal of any member would induce the inactivity of its former party. Conditions (I) and (II) jointly lead to the following lemma.

Lemma 3. *In any two-party equilibrium, both parties are efficient.*

Lemma 3 implies that, in every equilibrium with two parties, each party member is pivotal: If some $i \in M^L$ would stop contributing to his party, he would cause

party L's inactivity and the certain implementation of platform r. As a result, each party has less then $C/c + 1$ members in any political equilibrium. Put differently, both member sets M^L and M^R are finite, and there are independent agents in any equilibrium. Intuitively, party L can be seen as a local public good from which all citizens with similar policy preferences benefit. As common with public goods, there is free-riding in equilibrium: although each citizen with an ideal point close to or left of platform l benefits from the activity of party L, he prefers to bear as little of the provision cost as possible himself.

Lemma 3 also implies that, in equilibrium, citizens do not join a party L to merely affect the choice of its platform l. If there were an equilibrium with a non-efficient party, then its members would only prefer to stay active if their exit would lead to a large change in platform l. Lemma 3 clarifies that this motive is never the (only) incentive for party membership in a two-party equilibrium.

The proof of Lemma 3 is by contradiction. For the basic idea, consider a potential equilibrium in which party L is not efficient and its platform is given by l_0. Then, the exit of the member with the most leftist ideal point would not cause party L's inactivity, but shift its party median and its platform to a more rightist position $l_1 > l_0$. The member is only willing to maintain his membership in L if the reduction in his policy payoff would be large enough to exceed the saved membership cost c. But if this were true, the entry of an independent citizen with a more rightist ideal point $w_i > l_1$ would lead to the same shift in the party median and the platform. Moreover, the policy payoff of this entrant would increase sufficiently to exceed the membership cost c as well, as I show in the formal appendix. Since an allocation with a non-efficient party hence cannot be *exit-stable* and *entry-stable* at the same time, it cannot represent a political equilibrium.

The next results restrict the set of supportable policy platforms (l, r) in two-party equilibria. To clarify the exposition, I again focus on the possible locations of the leftist part's platform l. I start with a property that follows from the *exit-stability* requirement (I). By the previous arguments, the exit of any members would cause L's inactivity and ensure the implementation of platform r. Hence, the policy gains of party L's activity must be large enough to outbalance the membership cost. In other words, an agent in M^L will only contribute to the provision the local public good if his private benefit is larger than his private costs. Formally, the platforms (l, r) in each political equilibrium must satisfy the condition

$$\Gamma(l, r) = p(l, r)(r - l) \geq c \,, \tag{6}$$

where the *policy effect function* Γ measures the policy gain for any agent with ideal

point $w_i < l$ (see previous section). Intuitively, the members of party L are only willing to support party L it it has a sufficiently large effect on the expected policy. This involves two aspects: First, the distance between the platforms l and r has to be large enough. Second, the winning probability $p(l, r)$ must be large enough to yield a sufficiently large effect on the expected policy. By both aspects, inequality (6) restricts the set of equilibrium platforms l and r considerably, as specified in the following lemma.

Lemma 4. *There is a unique number $r_0 < c$ such that, in a two-party equilibrium, party L can only be exit-stable if*

(i) platform r is located weakly to the right of r_0 and,

(ii) for any $r \geq r_0$, platform l is located in a uniquely defined interval $\left[\underline{b}(r), \bar{b}(r)\right]$ such that $\bar{b}(r) < r$ and $\underline{b}(r) \leq \bar{b}(r)$ (with strict inequality for $r > r_0$).

Lemma 4 provides two insights that follow from condition (6). Part (i) implies that both sides of the political spectrum have to be represented in every two-party equilibrium. Put differently, there are no two-party equilibria in which both parties run with leftist platforms below the threshold r_0. Intuitively, if one party already commits to a leftist platform by selecting an appropriate candidate, then an even more leftist party cannot provide sufficient policy gains to any citizen. In particular, the more leftist party will either make no difference at all or have so little electoral prospects that it is unable to find sufficient support in the population.

Part (ii) of Lemma 4 clarifies that, even though both parties are formed endogenously, the supportable platforms of both parties are strategically interdependent. In particular, party L can only find sufficient support in the population to run against party R with platform r, if it selects a candidate with an ideal point in the interval $\left[\underline{b}(r), \bar{b}(r)\right]$. For any other candidate, no citizens would be willing to engage politically, either because the difference between both parties is too small or because the electoral prospects of party L are too poor. Importantly, the location of these bounds does not depend on the details of the member set M^L. The composition of M^L however determines, first, which platforms in the relevant interval are available and, second, which of the available platforms in this interval is selected.

The final property follows from the *entry-stability* requirement (II), according to which independent citizens must not have an incentive to join an active party $Q \in \{L, r\}$. The following arguments are specific to a model with endogenous party formation and, to the best of my knowledge, have not been made in the previous literature. Entering party L allows an independent citizen to potentially affect platform l and increase his policy payoff, but it also comes with the membership

cost c. The requirement of *entry-robustness* restricts the set of equilibrium platforms (l, r) in the following way.

Lemma 5. *Consider some (l, r) such that condition (6) holds. There are a unique number $\tilde{r} \in (r_0, 2c)$ and a function $\tilde{b} : [\tilde{r}, \infty) \to \mathbb{R}$ such that party L can only be entry-stable in a two-party equilibrium if*

1. *either $r \leq \tilde{r}$,*
2. *or $r > \tilde{r}$ and $l \geq \tilde{b}(r)$, with $\tilde{b}(r) \in \left[\underline{b}(r), \bar{b}(r)\right]$.*

Lemma 5 provides a lower bound on the platform l, conditional on platform r being large enough. Importantly, this condition rules out some allocations that are *exit-robust*, i.e., that satisfy condition (6). Specifically, any allocation with platforms $r > \tilde{r}$ and l between $\underline{b}(r)$ and $\tilde{b}(r)$ is *exit-robust*, but not *entry-robust*.

For the intuition behind this result, consider an allocation in which both platforms l and r are relatively extreme, and a moderate independent citizen with ideal point $w_i \in (l, r)$.[22] Assume that party L's effect on the expected policy with platform l is smaller than the effect it could achieve with platform w_i, $\Gamma(w_i, r) > \Gamma(l, r)$. Then, if citizen i would enter party L, he would be selected as candidate and benefit from a policy gain. If the difference between l and w_i is large enough, this policy gain exceeds the membership cost c: The initial allocation cannot be an equilibrium. The possibility to recruit more moderate citizens hence rules out equilibria in which both platforms are too extreme.

The formal proof of Lemma 5 involves three steps. First, an independent agent i with ideal point $w_i > l$ can achieve the policy gain

$$\tilde{v}_i(w_i, r) - \tilde{v}_i(l, r) = \Gamma(w_i, r) + p(l, r)\,(2w_i - l - r) \tag{7}$$

by entering party L and becoming its candidate. As I show in the appendix, the incentives to join part L satisfy the Gans & Smart (1996) single-crossing property: If the policy gain from entering party L is large enough to exceed c for an agent with some ideal point w_i, then the same is true for agents with more moderate ideal points. Second, the independent agent i will actually be selected as candidate of party L if and only if his ideal point w_i achieves a higher policy effect than platform l. To verify *entry-stability*, it is hence sufficient to check whether joining L is profitable for the agent with the most moderate ideal point w_i that still ensures $\Gamma(w_i, r) \geq \Gamma(l, r)$. Third, I concentrate on the set of platform pairs such that $l = \underline{b}(r)$, i.e., that l is just

[22] By Lemma 3, the set of party members is finite in each equilibrium. Hence, the existence of an independent agent with appropriate ideal point is ensured.

large enough to ensure *exit-stability*. Within this set of platform pairs, the policy gain to an independent agent with the before-defined ideal point is monotonically increasing in r. This finally allows me to show that entering party L is profitable for some independent agents if and only if both platforms are sufficiently extreme with $r > \tilde{r}$ and $l < \tilde{b}(r)$.

Combining Lemmas 4 and 5, we know that for any platform $r > r_0$, party L can only have a stable member set M^L if its platform is located between the upper bound $\bar{b}(r)$ and a lower bound given by

$$\hat{b}(r) \coloneqq \begin{cases} \underline{b}(r) & \text{for} \quad r \in [r_0, \tilde{r}] \ ; \\ \tilde{b}(r) & \text{for} \quad r > \tilde{r} \ . \end{cases} \tag{8}$$

As the game is symmetric between both political parties, the platforms l and r must satisfy corresponding conditions to ensure that party R can be both *exit-stable* and *entry-stable*. Note that Lemmas 4 and 5 provide necessary conditions for the platform pair (l, r) to be supported in equilibrium. The following proposition shows that these conditions are also jointly sufficient for the existence of a political equilibrium with platforms l and r locations. More precisely, there exist an equilibrium with some stable membership structure (M^L, M^R) and platforms (l, r) whenever the latter satisfy the following conditions.

Proposition 1. *There exists a two-party equilibrium with platforms* (l, r) *if and only if*

(i) $\min\{-l, r\} \geq r_0$,

(ii) $l \in \left[\hat{b}(r), \bar{b}(r)\right]$, *and*

(iii) $r \in \left[-\bar{b}(-l), -\hat{b}(-l)\right]$.

Proposition 1 provides a complete characterization of the set of the policy platforms l and r that can be supported by a stable membership structure (M^L, M^R). It is important to note that, for each specific (stable) member set M^L, platform l is uniquely determined by some reaction function $l^*(r)$ (see Lemma 2). The interval $[\hat{b}(r), \bar{b}(r)]$ can be regarded as the collection of all reaction functions over the complete set of stable membership structures. The set of supportable platforms (l, r) is given by the intersection of the entire set of reaction functions for both parties L and R. Figure B in Appendix B depicts these collections and their intersection graphically for a numerical example with a normally distributed median voter position.

A classical question in the economic theory of political competition refers to the degree of convergence or divergence between the parties' equilibrium platforms. Most famously, the Downs (1957) model has a unique equilibrium with full convergence at the median voter's preferred policy. In the citizen candidate model, in contrast, there is policy divergence in every political equilibrium with two competing candidates (Osborne & Slivinski 1996 and Besley & Coate 1997). In the following, I revisit this question in a setting with endogenous party formation, distinguishing between symmetric and asymmetric equilibria.[23]

Proposition 2. *The platform distance $r - l$ in two-party equilibria is bound from below and from above. In particular,*

(a) there exists a symmetric equilibrium with platforms $r = -l$ if and only if $r \in [c, \bar{r}]$, where $\bar{r} \in (c, \infty)$ is the unique fixed point of function $-\tilde{b}$;

(b) in every asymmetric equilibrium with platforms $r \neq -l$, the platform distance is

(i) strictly larger than $2c$, and

(ii) strictly smaller than $\bar{d} \coloneqq 2\bar{r} - \frac{2}{3}\left[\tilde{r} + \tilde{b}(\tilde{r})\right] > 2\bar{r}$.

Proposition 2 establishes the main result of this paper: In all political equilibria, the platform distance $r - l$ is bound from below by $2c$ and from above by $\bar{d} > 2c$. First, policy convergence is limited by the cost of political activity c: both parties will only be supported by party members if their platforms are sufficiently different. This result is closely related to the one in the citizen candidate models by Osborne & Slivinski (1996) and Besley & Coate (1997). Second, policy divergence (polarization) is also limited, in contrast to the citizen candidate model. In particular, the upper bound on the platform distance $r - l$ depends on the fixed point of function $\tilde{b}$, which was derived from the requirement of *entry-stability*. Hence, equilibria with too divergent platforms are ruled out by the option to recruit a moderate candidate from the set of independent citizens, thereby committing to a moderate platform. Intuitively, the necessity to ensure the support of party members induces a centrifugal force, while the party member's desire to coordinate on a candidate with good electoral prospects induces a centripetal force.

The formal arguments behind the limited convergence is straightforward. The winning probability of at least one competing party has to be given by $1/2$ or less in each equilibrium. Unless the platform distance is at least $2c$, this party would not have a sufficient effect on the expected policy to make any citizen willing to

[23] While Proposition 1 already provides implicit insights on the distance between supportable policy platforms, Proposition 2 delivers explicit results.

support it. The proof for the limited divergence is more complicated. It involves two steps. First, I use implicit differentiation to show that function $\tilde{b}$ has a unique fixed point, which limits the platform distance in symmetric equilibria. Second, I derive upper and lower bounds on the slope of the functions $\tilde{b}$ and $\underline{b}$, which can be used to identify the upper bound $\bar{d}$ in asymmetric equilibria.

6.2 Political equilibria with one uncontested party

In the following, I derive two results on the set of political equilibria with only one active party $Q \in \{L, r\}$. In such an allocation, the incentives for political activity in an allocation are substantially different from those in an allocation with two active parties. With two active parties, the agents' optimal choices depend strongly on whether their behavior can improve the electoral prospects of one party, relative to the competing party. With one active party, potential party members are mainly interested in affecting decision-making within their party. The extent to which a single agent can affect candidate selection depends on the party's cohesion, i.e., the similarity of the agents in M^Q with respect to their policy preferences. To account for this cohesion, I denote by m_+^Q the ideal point of the party member $i \in M^Q$ that is adjacent to and weakly larger than m^Q, and by m_-^Q the ideal point of the member that is adjacent to and weakly smaller than m^Q.

Lemma 6. *In any one-party equilibrium, one of the following conditions is satisfied:*

(i) Party Q is efficient and M^Q satisfies $\max\left\{m_+^Q - m^Q, m^Q - m_-^Q\right\} < 2c$.

(ii) Member set M^Q satisfies $m_+^Q - m^Q = m^Q - m_-^Q = 2c$.

Part (i) shows that, in one type of equilibria with an uncontested party, this party has to be efficient, i.e., the exit of each party member would cause its inactivity.[24] Additionally, the member set M^Q has to be sufficiently coherent in each equilibrium, measured by the distance between the ideal points of the median party member and the most similar party members, m_+^Q and m_-^Q. By part (ii), however, there are additional equilibria in which the only active party is not efficient and its member set M^Q exhibits a knife-edge degree of cohesion. These necessary properties follow from the requirements of *entry-stability* and *exit-stability*.

First, an independent agent could profit from joining party Q if he could affect the party platform q (which equals the implemented policy) to a sufficiently large degree. In particular, his policy gain from the change in q would have to be large enough to exceed the membership cost c. As shown in Lemma 1, an uncontested

[24] Recall that this was a necessary condition for two-party equilibria (see Lemma 3).

party always selects its party median as candidate. Hence, *entry-stability* requires that the party median m^Q must only move a little when a single agent enters M^Q. Specifically, the distance between m^Q and the adjacent ideal points m_+^Q and m_-^Q has to be weakly smaller than $2c$.

Second, a party member could profit from leaving party Q if the resulting loss in his policy payoff is smaller than the membership cost c. This is ruled out if party Q is efficient as required in part (i): his exit induces the party's inactivity and hinders the implementation of any policy, implying a policy payoff of $-\infty$. In the knife-edge case (ii), the exit of any party member affects the location of the party median m^Q just as much that the resulting policy loss equals c. Hence, all members of party Q are indifferent between leaving q and staying.

The second result clarifies the conditions under which one-party equilibria exist and the set of supportable platforms in these equilibria. It follows from another aspect of *entry-stability*: the foundation of a new party by some independent agent. If there is only one active party Q, then any independent agent i could choose to enter the other party $-Q$ and enable it to run in the general election by contributing the entire cost of running, $\alpha_i^{-Q} \geq C$. In a one-party equilibrium, this deviation must not be profitable for any independent agent $i \in I$.

Proposition 3. *If there is a number $z > 2C$ such that $z\Phi(-z/2) > C$, then there is no one-party equilibrium. Otherwise, there is a number $r_1 \in [0, C)$ such that there is a one-party equilibrium with platform q if and only if $q \in [-r_1, r_1]$.*

By Proposition 3, equilibria with an uncontested active party exist if and only if the electoral risk implied by Φ is small enough given the cost of running C or, vice versa, the cost C is large enough given distribution Φ. If such equilibria exist, the platform of the uncontested party has to be sufficiently close to the expected median voter position 0. The economic intuition behind both parts of this result is that the cost of running C works as a barrier to market entry, i.e., the formation of a competing party. An independent citizen can only benefit from setting up a new party to challenge Q if the expected effect on the implemented policy x is large enough to cover C. A more moderate platform q and a smaller level of electoral risk imply that the electoral prospects of potential entrants are limited, deterring independent citizens from challenging party Q.

For the logic behind the first sentence of Proposition 3, consider an allocation in which the uncontested party's platform q equals the expected median voter position 0. If an independent agent i starts a new party that runs with platform $w_i < -2C$ against party Q, its winning probability is given by $p(w_i, 0) = \Phi(w_i/2) < 1/2$. If electoral risk is large, this winning probability is sufficiently close to $1/2$ so that

the resulting policy gain exceeds C: agent i benefits from starting a new party and entering political competition. Hence, there is neither an equilibrium in which the uncontested party's platform is given by the expected median voter position, nor an equilibrium with any other, even less competitive platform. If the electoral risk is small, in contrast, the winning probability of a new party with any platform $w_i < -2C$ is too limited to make political activity profitable. By continuity, the same is true if platform q is located sufficiently close to 0.

7 Comparative statics and limit results

In the previous sections, I have identified the set of policy platforms that can arise in political equilibria with one and two active parties, given some fixed median voter distribution Φ, membership cost c and cost of running C. This section investigates the effects of changes in the cost parameter c and in the degree of electoral risk implied by Φ. Additionally, it studies the set of equilibrium platforms for the limit cases of costless party membership and electoral certainty.

The first set of results focuses on two-party equilibria, in particular on the comparative static effects on the lower and upper bounds on the platform distance $r-l$. I start by considering variations in the membership cost c.

Proposition 4. *An increase in the membership cost c leads to a strictly larger*

(i) minimum platform distance $2c$ and

(ii) maximum platform distance $2\bar{r}$ in symmetric equilibria.

For the limit case $c = 0$, the platform distance in any equilibrium is bounded by 0 from below and by $\lim_{\sigma\to 0} 2\bar{r} = 1/\phi(0) > 0$ from above.

Consider first the lower bound on the platform distance, $r - l \geq 2c$. In equilibrium, party members are only willing to maintain their activity if each party's activity has a sufficiently large effect on expected policy, i.e., if the platform distance is large enough. As the cost of political activity c becomes larger, party members demand increasing policy effects and platform distances to maintain their political engagement. If, however, the membership cost approaches zero, the members get willing to accept more closer platforms. In the limit, party membership is costless and is even consistent with full policy convergence.

With respect to the most divergent equilibria, increasing membership costs tighten the combined coordination and free-riding problem faced by potential activists. Consider an independent agent i whose ideal policy $w_i \in (l, r)$ would increase the

policy effect of party L, relative to platform l. As long as the difference between the policy effects of w_i and l is not large enough to exceed c, the independent agent prefers to free-ride on the current members of party L. With an increasing membership cost c, an even larger difference between w_i and l is required to make joining party L profitable. Hence, more extreme platforms by both parties can be supported in a two-party equilibrium.

If the membership cost converges to zero, in contrast, this coordination problem vanishes: an independent agent is willing to join party L whenever his ideal point w_i allows to achieve a larger policy effect than l. Put differently, the median party member is always able to recruit his preferred candidate and to choose his preferred policy platform. Consider a case where both median party members have extreme policy preferences, $m^L \to -\infty$ and $m^R \to \infty$. Then, each party $Q \in \{L, R\}$ selects the platform that maximizes the policy effect $\Gamma(q, -q)$, given its opponent's platform $-q$. These mutually best responses are given by $l = -\left[2\phi(0)\right]^{-1}$ and $r = \left[2\phi(0)\right]^{-1}$, respectively, as I show in the appendix.

Next, I study the effect of variations in the electoral risk as implied by the distribution Φ. For this purpose, I restrict my attention to distributions within some family that satisfy the following assumption.

Assumption 2. *The probability distribution Φ belongs to a family of distribution functions $(\Phi_\sigma)_{\sigma\in\mathbb{R}_+}$ such that, for any fixed $x < 0$,*

(i) $\frac{d\Phi_\sigma(x)}{d\sigma} > 0$,

(ii) $\lim_{\sigma\to 0} \Phi_\sigma(x) = 0$, *and*

(iii) $\lim_{\sigma\to\infty} \Phi_\sigma(x) = 1/2$.

If Assumption 2 is satisfied, all members of family $(\Phi_\sigma)_{\sigma\in\mathbb{R}_+}$ can be ordered with respect to the implied electoral risk. In particular, Φ_σ is a mean-preserving spread of $\Phi_{\sigma'}$ if and only if $\sigma > \sigma'$. For example, this assumption is satisfied by the families of normal distributions with mean 0, logistic distributions and Laplace distributions. Treating σ as a parameter that measures the degree of electoral risk, I can derive the following result.

Proposition 5. *An increase in the degree of electoral risk σ*

(i) has no effect on the minimum platform distance $2c$, and

(ii) strictly increases the maximum platform distance $2\bar{r}$ in symmetric equilibria.

For the limit case $\sigma \to 0$, the party platforms are given by $(l, r) = (-c, c)$ in every two-party equilibrium.

By Proposition 5, the upper bound of the platform distance $r - l$ is increasing in the degree of electoral risk, while the lower bound remains constant. Intuitively, the more uncertain the outcome of an election is, the more attractive a party median finds his own ideal point m^L compared to the ideal point $w_i \in (l, r)$ of a more moderate citizen. Hence, extreme party medians become less interested in recruiting a moderate citizen and selecting him as party candidate. Hence, an increase in electoral risk implies that more divergent platforms can be supported in two-party equilibria.

In the limit case of full electoral certainty, $\sigma \to 0$, the location of the population median m is perfectly known. In this case, the members of each party have a very strong incentive to choose a platform that is closer to the median voter than the competing party. This eliminates all symmetric equilibria as well as all symmetric equilibria with a platform distance $r - l > 2c$. Consequently, only the platforms $l = -c$ and $r = c$ can be supported by stable parties in a two-party equilibrium. Given these platforms, the membership cost c deters each moderate citizen with ideal point $w_i \in (-c, c)$ from entering one of the parties.[25]

Note that the difference between models with and without endogenous party formation becomes most obvious in this limit case, on which the basic literature on the citizen candidate focuses (Osborne & Slivinski 1996 and Besley & Coate 1997). In both models as in the model studied here, costs of political activity give rise to a lower bound on the platform distance in two-candidate equilibria. Without independent citizen-candidates, however, there is no mechanism that limits policy polarization in equilibrium. Consequently, there is a large multiplicity of two-party equilibria with different platform distances. This well-known weakness of the citizen candidate model contrasts sharply with the unique pair of equilibrium platforms established in Proposition 5.

A result of special interest can be derived for the twofold limit case, where party membership is costless and there is full electoral certainty. This is the only case for which every two-party equilibrium involves full convergence of both party platforms at the median voter position.

Corollary 1. *If and only if both $\sigma \to 0$ and $c \to 0$, both party platforms l and r equal the median voters' ideal policy in every two-party equilibrium.*

This result confirms for a special case the famous Downs (1957) result, full convergence of platforms. Arguably, both conditions (zero membership costs, no

[25] Given an allocation with more divergent symmetric platforms, moderate citizens with ideal points in (l, r) would profit from joining one of the parties. Given an allocation with more convergent platforms, the members of at least one party would profit from leaving their party and causing its inactivity.

electoral risk) seems very restrictive from an applied perspective. The basic message of Corollary 1 and the previous results is the following: In any political competition between endogenously formed parties, there is a centripetal force that pushes the competing parties' platforms to converge towards the median voter. Full convergence is not a robust prediction, however, but only a natural limit case that results if all kinds of frictions (costs of activity, limited information) vanish. It should be noted that this convergence result can never be confirmed in the citizen candidate model. In the corresponding limit case with zero cost of running and full electoral certainty, there continues to exist a multiplicity of equilibria with two competing candidates. This includes equilibria where both candidates share the median voter's ideal point. It also includes equilibria where the candidates have (strongly) divergent ideal points (see Besley & Coate 1997, Osborne & Slivinski 1996).

Finally, I provide two results on the set of political equilibria with one uncontested party. The first one refines Proposition 3 by clarifying that one-party equilibria exist if and only if the degree of electoral risk is sufficiently small.

Proposition 6. *There is a threshold $\sigma_1 > 0$ such that,*

(i) for $\sigma > \sigma_1$, there is no one-party equilibrium;

(ii) for $\sigma = \sigma_1$, the platform $q \in \{l, r\}$ in every one-party equilibrium is given by the median voter's ideal point 0;

(iii) for any $\sigma < \sigma_1$, there is a number $r_1 \in (0, C)$ such that there is a one-party equilibrium with platform q if and only if $q \in [-r_1, r_1]$. For $\sigma \to 0$, r_1 converges to $C/2$.

The final result compares the implemented policies in two-party and one-party equilibria for some strictly positive values of the membership cost c and the cost of running C (focusing on the case of full electoral certainty). It seems natural to ask whether two competing parties cater more or less to the preferences of the (decisive) median voter than a single, uncontested party.[26] In the citizen candidate model by Besley & Coate (1997), the implemented policy in any one-candidate equilibrium is ex post closer to median voter's ideal point than in the unique two-candidate equilibrium. With endogenous party formation, I come to a different conclusion.

Corollary 2. *In the limit case $\sigma \to 0$, the median voter is ex post strictly worse off in some one-party equilibria than in every two-party equilibrium.*

[26] For a symmetrical distribution of ideal points in the population, this is equivalent to asking whether social welfare (i.e., the integral over all citizens' policy payoffs) is larger or smaller in two-party equilibria than in one-party equilibria.

The Corollary follows directly from Propositions 5 and 6, which provide sharp results for the limit case of full electoral certainty. For the intuition behind it, consider a party whose members have very coherent, but extreme policy preferences (in the sense of differing strongly from the median voter's preferences). If this is the only active party, it is relatively complicated for moderate independent citizens to stand up against this party and affect political decisions. If there are two competing parties, then moderate independents can more easily become active by joining or supporting the less extreme party. Loosely speaking, political competition reduces the barriers to political activity, working as a safeguard against political extremism. In the model, this idea is represented by the difference between the cost of party membership c and the cost of setting up a competitive party $C > 2c$. Altogether, Corollary 2 conflicts with the results of the citizen candidate model, but confirms the conventional view that political competition is beneficial, ensuring that politicians respect the voters' interest.[27]

8 Conclusion

This paper has investigated electoral competition between endogenously formed parties in a new model that arguably brings theory closer to real-world politics. The analysis has focused on the set of policy platforms that can be offered by stable political parties and on the properties of their membership structures. In particular, I have derived the implications of *entry-stability* and *exit-stability* on the policy platforms in political equilibria with two competing parties and with one uncontested party. I have provided two main results. First, I have shown that the platform distance in two-party equilibria is always strictly positive, but limited. This result is in contrast to the classical Downs (1957) result of full policy convergence, which fails to comply with empirical observations. It is also in contrast to the results of the citizen candidate models by Osborne & Slivinski (1996) and Besley & Coate (1997), in which there is a multiplicity of two-party equilibria with indeterminate platform distance. The difference can be seen most obviously in the benchmark case of full electoral certainty, where both parties' platforms are uniquely determined in the party formation model only. Second, I have shown that the implemented policy can differ more strongly from the median voter's preferences in equilibria with one uncontested party than in equilibria with two competing parties. Hence, multi-

[27] Proposition 6 implies that this result extends to cases with a low level of electoral risk as measured by σ. For higher levels of σ, the set of supportable platforms in one-party equilibria is closer to the expected median voter position. For even higher levels of electoral risk, there exist no equilibria with one uncontested party.

party competition can be seen as a safeguard against political extremism. Again, this result is in contrast to the findings in the citizen candidate model.

For the sake of clarity, the analysis of this paper has focused on a simple environment, including an abstract one-dimensional policy space. The model is however tractable enough to study more complex policy decisions, especially in the commonly studied benchmark case of full electoral certainty. For example, it could be used to investigate political competition over linear income taxation as in Dixit & Londregan (1998) or non-linear income taxation as in Brett & Weymark (2017). A richer model could also allow for, e.g., a larger number of potential parties, more general rules with respect to intra-party decision-making, more general policy preferences, or a different modeling of electoral uncertainty.

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Appendix

A Proofs

Proof of Lemma 1

Proof. Given $\hat{r} = \emptyset$, the members of party L expect their candidate to become president and choose policy x with certainty. Because candidate j will implement his ideal point w_j, choosing a candidate from M^L is equivalent to choosing a policy from the set of the member's ideal points. The policy preferences are single-peaked by (1). By standard arguments, this single-peakedness implies that voting is monotonic in every election with two alternatives x_1 and x_2 in X, and that the ideal point of the median party member is a Condorcet winner in the set of all member's ideal points. Hence, the median party member prevails in the primary's pairwise elections. □

Proof of Lemma 2

Lemma 2 identifies the optimal choice of party platform l in the primary election of party L, conditional on the membership structure M^L and belief $\hat{r}$. It is proven through a series of lemmas.

Lemma 7. *Given any platform belief $\hat{r}$, the platform preferences of party members over the set of potential platforms satisfy a version of the single-crossing property by Gans & Smart (1996).*

Proof. Under the single-crossing property proposed by Gans & Smart (1996), the preferences of agent i with respect to any pairwise comparisons between two alternatives are monotonic in his bliss point ω_i. Consider two alternatives l_1 and l_2 for party L's platform such that $l_1 < l_2 < \hat{r}$. An agent with bliss point w_i prefers l_1 to l_2 if and only if

$$\begin{aligned} F(l_1, l_2, \hat{r}, w_i) &= \tilde{v}_i(l_1, \hat{r}) - \tilde{v}_i(l_2, \hat{r}) \\ &= p(l_1, \hat{r})(|w_i - \hat{r}| - |w_i - l_1|) - p(l_2, \hat{r})(|w_i - \hat{r}| - |w_i - l_2|) > 0 \end{aligned}$$

The derivative of function F with respect to w_i is given by

$$\frac{dF(\ )}{dw_i} = \begin{cases} 0 & \text{for} \quad w_i \leq l_1 \\ -2p(l_1, \hat{r}) < 0 & \text{for} \quad w_i \in (l_1, l_2] \\ 2\left[p(l_2, \hat{r}) - p(l_1, \hat{r})\right] > 0 & \text{for} \quad w_i \in (l_2, \hat{r}] \\ 0 & \text{for} \quad w_i \geq \hat{r} \end{cases}$$

For $\Gamma(l_1, \hat{r}) > \Gamma(l_2, \hat{r})$, F has a unique cutoff $\hat{w}$ in the interval (l_1, l_2). All agents with

$w_i \leq l_1$ prefer l_1 and we get the following version of the single-crossing property:

$$F(l_1, l_2, \hat{r}, w_i) \leq 0 \Rightarrow F(l_1, l_2, \hat{r}, w_j) < 0 \ \forall \ w_j > w_i, \text{ and}$$
$$F(l_1, l_2, \hat{r}, w_i) \geq 0 \Rightarrow F(l_1, l_2, \hat{r}, w_k) > 0 \ \forall \ w_k < w_i$$

For $\Gamma(l_1, \hat{r}) < \Gamma(l_2, \hat{r})$, F has a unique cutoff $\hat{w}$ in the interval $(l_2, \hat{r})$. In this case, all agents with $w_i \leq l_1$ prefer platform l_2 and the preferences exhibit the following monotonicity:

$$F(l_1, l_2, \hat{r}, w_i) \leq 0 \Rightarrow F(l_1, l_2, \hat{r}, w_j) < 0 \ \forall \ w_j < w_i, \text{ and,}$$
$$F(l_1, l_2, \hat{r}, w_i) \geq 0 \Rightarrow F(l_1, l_2, \hat{r}, w_k) > 0 \ \forall \ w_k > w_i$$

For $\Gamma(l_1, \hat{r}) = \Gamma(l_2, \hat{r})$, all agents with $w_i \leq l_1$ as well as $w_i \geq \hat{r}$ are indifferent between both platforms, while all agents with $w_i \in (l_1, \hat{r})$ strictly prefer the moderate platform l_2. Trivially, the preferences satisfy the single-crossing property in the following sense:

$$F(l_1, l_2, \hat{r}, w_i) \geq 0 \Rightarrow F(l_1, l_2, \hat{r}, w_k) \geq 0 \ \forall \ w_k \in \mathbb{R}$$

Similar arguments apply for other constellations, e. g. $l_1 < \hat{r} < l_2$. □

Lemma 8. *For any member set M^L and platform belief $\hat{r}$, there is a Condorcet winner in the primary election of party L.*

Proof. Let the finite set of feasible platforms, i.e., the set of bliss points of party L's members, be given by A. Denote by l^* the platform in A that maximizes the utility of the median party member with platform $w_i = m^L$:

$$l^* = \arg\max_{l \in A} \tilde{v}_i(l, \hat{r}) = -p(l, \hat{r}) \left| \hat{r} - m^L \right| - [1 - p(l, \hat{r})] \left| l - m^L \right|$$

By the single-crossing property established in Lemma 7, platform l^* is preferred by a majority of party members (the median member plus either all members with $w_j \leq m^L$ or all members with $w_j \geq m^L$) to any other available platform $l' \in A$. Consequently, l^* wins any pairwise election and represents a Condorcet winner. □

Lemma 9. *For any $r \in \mathbb{R}$, function $\Gamma(l, r) \coloneqq p(l, r)(r - l)$ has a unique maximizer $l_\Gamma(r) \in (-\infty, r)$ and is strictly quasi-concave in its first argument for $l \in (-\infty, r)$. For r below (above) $[2\ \phi(0)]^{-1}$, $l_\Gamma(r)$ is below (above) $-r$.*

Proof. Fix some $r \in \mathbb{R}$. As Φ is assumed to be continuously differentiable, the same is true for Γ. For any $l \leq r$, the first two derivatives of $\Gamma(l, r)$ with respect to l are given by

$$\Gamma_1(l, r) = p'(l, r)(r - l) - p(l, r) \text{ and}$$
$$\Gamma_{11}(l, r) = p''(l, r)(r - l) - 2p'(l, r) \ ,$$

where $p'(l,r) = 1/2\phi((l+r)/2)$ and $p''(l,r) = 1/4\phi'((l+r)/2)$. At any extremum of Γ in l, the second derivative is given by

$$\begin{aligned}\Gamma_{11}(l,r) &= p''(l,r)\frac{p(l,r)}{p'(l,r)} - 2p'(l,r) = p(l,r)\left[\frac{p''(l,r)}{p'(l,r)} - 2\frac{p'(l,r)}{p(l,r)}\right] \\ &= p(l,r)\left[\frac{1}{2}\frac{\phi'(z)}{\phi(z)} - \frac{\phi(z)}{\Phi(z)}\right] < 0\,,\end{aligned}$$

where $z \coloneqq (l+r)/2$. By the log-concavity imposed in Assumption 1, the term in brackets is strictly negative for any $z \in \mathbb{R}$. Consequently, Γ is strictly quasi-concave, i.e., it has at most one local maximum and no local minimum in the range $l \in (\infty, r)$.

To show the existence of a local maximum, I consider the limit of $\Gamma(l,r)$ for l converging to $-\infty$. For any $r \in \mathbb{R}$, this is given by

$$\lim_{l\to-\infty}\Gamma(l,r) = \lim_{l\to-\infty}\frac{r-l}{\frac{1}{\Phi(z)}} = \lim_{l\to-\infty}\frac{1}{\frac{\phi(z)}{2\Phi(z)^2}} = \lim_{l\to-5\infty} 2\frac{\Phi(z)^2}{\phi(z)} = 0$$

The last equality sign follows because $\lim_{l\to-\infty}\Phi(z) = 0$ and $\Phi(z)/\phi(z) \in (0,a)$ for any $z < 0$ by Assumption 1, where $a \coloneqq \Phi(0)/\phi(0) = [2\phi(0)]^{-1}$ is given by some finite number. Hence, $\Gamma(l,r)$ converges to zero for $l \to -\infty$. Moreover, $\Gamma(l,r)$ is weakly negative for all $l \geq r$, and strictly positive for all $l \in (-\infty, r)$. Consequently, Γ has a unique maximizer $l_\Gamma(r) \in (-\infty, r]$ for any $r \in \mathbb{R}$.

Finally, for any $l = -r < 0$, $\Gamma_1(-r,r) = p_1(-r,r)\ (r+r) - p(-r,r) = r\ \phi(0) - \frac{1}{2}$. If and only if r is larger than (smaller than) $[2\ \phi(0)]^{-1}$, this derivative is positive (negative), ensuring that $l_\Gamma(r)$ is strictly larger (smaller) than $-r$. □

Lemma 10. *For any set of potential platforms A and belief $\hat{r} > m^L$, the policy payoff of the median party member is maximized by platform* $\max\left\{m^L, \lambda'(\hat{r}, A)\right\}$*, where* $\lambda'(\hat{r}, A) = \arg\max_{l\in A}\Gamma(l,\hat{r})$.

Proof. Let the set of party L's potential platforms be given by A, and assume that m^L is an element of A. Given platform l, the median member's expected policy payoff is $\tilde{v}_{m^L}(l,\hat{r}) = p(l,\hat{r})\left(\left|\hat{r}-m^L\right| - \left|l-m^L\right|\right) - \left|\hat{r}-m^L\right|$.

First, for any $l < m^L$, we have $\tilde{v}_{m^L}(m^L,\hat{r}) - \tilde{v}_{m^L}(l,\hat{r}) = \left[p(m^L,\hat{r}) - p(l,\hat{r})\right](\hat{r}-m^L) + p(l,\hat{r})(m^L-l) > 0$. Hence, the median member strictly prefers his own ideal point m^L to any more extreme platform.

Second, for any $l \in \left[m^L, \hat{r}\right]$, the party median's policy payoff simplifies to $\tilde{v}_{m^L}(l,\hat{r}) = \Gamma(l,\hat{r})$. This directly implies that the party median will prefer the platform that maximizes the policy effect $\Gamma(l,\hat{r})$ over the elements in A that are weakly larger than m^L. Formally, this platform is given by $\lambda'(\hat{r}, A)$ if and only if this is weakly larger than m^L. In the opposite case $\lambda'(\hat{r}, A) < m^l$, the quasi-concavity of Γ ensures that m^L has a larger policy effect than any more moderate platform. Lemma 2 follows directly for A being equal to the set of ideal points of all members in M^L. □

Proof of Lemma 3

Proof. Assume there is a two-party equilibrium with membership structures M_0^L, M_0^R, party medians m_0^L, m_0^R and platforms $l_0 = l(M_0^L, r_0)$, $r_0 = r(M_0^R, l_0)$ such that party L is not efficient, i.e., $\sum_{i \in N} \alpha_i^L \geq C + c$. This requires that neither a member $j \in M^L$ nor an independent citizen must have an incentive to change his contribution α_i^L and, potentially, change his membership status in party L. In the following, I show that this gives rise to a contradiction.

First, if some member i contributes more than $\alpha_i^L = c$ or that some independent agent j contributes $\alpha_j^L \in (0, c)$, he could reduce his contribution without affecting the party L's platform l or party L's activity. This would clearly constitute a profitable deviation. Hence, there can only be an inefficient party in equilibrium if all members contribute exactly $\alpha_i^L = c$, while all independent agents provide $\alpha_j^L = 0$.

Second, if the most extreme member k of party L (the member with the lowest ideal point) reduces his contribution from c to 0, the party median increases to $m_1^L \geq m_0^L$ and the platform increases to $l_1 \geq l_0$. Because $\sum_{i \in N \setminus \{k\}} \alpha_i^L \geq C$, party L remains active. This deviation is profitable for k unless

$$\tilde{v}_k(l_1, r_0) - \tilde{v}_k(l_0, r_0) + c = \Gamma(l_1, r_0) - \Gamma(l_0, r_0) + c \leq 0\,.$$

Third, if an independent citizen j with bliss point $w_j \in (l_1, r_0)$ starts to contribute $\alpha_j^L = c$ and joins party L, the party median again increases to $m_1^L \geq m_0^L$. The platform either increases to $l_1 \geq l_0$ or to $w_j > l_1$. In particular, the latter can occur if and only if $\Gamma(w_j, r_0) \geq \Gamma(l_1, r_0)$. If the platform changes to l_1, the deviation is profitable for entrant j if and only if

$$\begin{aligned}\tilde{v}_k(l_1, r_0) - c - \tilde{v}_k(l_0, r_0) =& p(l_1, r_0)\,[r_0 + l_1 - 2w_k] - p(l_0, r_0)\,[r_0 + l_0 - 2w_k] - c \\ =& -\left[\Gamma(l_1, r_0) - \Gamma(l_0, r_0) + c\right] + 2(r - w_k)[p(l_1, r_0) - p(l_0, r_0)]\end{aligned}$$

is strictly positive. Note that the term in brackets is weakly negative if there is no profitable deviation for member k, and that the remaining term is strictly positive for all $\omega_k < r$ and $l_1 > l_0$. If the platform changes to w_j instead of l_1, the deviation is even more profitable for entrant j. Hence, in every potential equilibrium where party L is not efficient, there is a profitable deviation either for the most extreme member of party L or for some independent citizen. We can conclude that there is no two-party equilibrium with inefficient parties.

□

Proof of Lemma 4

Proof. First, assume there is an equilibrium with platforms l and r. By Lemma 3, parties are efficient in every political equilibrium. Hence, party L would become inactive and

platform r would be implemented with certainty if some member leaves party L and saves the contribution $\alpha_i^L \geq c$. For any member with ideal point weakly below the platform, $w_i \leq l$, this would be profitable if

$$v_i(r) - (\tilde{v}_i(l,r) - \alpha_i^L) \geq v_i(r) - \tilde{v}_i(l,r) + c = -\Gamma(l,r) + c > 0 \, .$$

Hence, inequality (6) must hold in every two-party equilibrium.

Second, I show that there is a number $r_0 < c$ such that, if and only if $r \geq r_0$, there exists a number $l \leq r$ such that condition (6) is be satisfied. By Lemma 9, Γ is strictly quasi-concave and has a unique maximizer $l_\Gamma(r) < r$ for any $r \in \mathbb{R}$. For $r \geq c$, we have $\Gamma(l_\Gamma(r), r) \geq \Gamma(-r, r) = r \geq c$. Hence, there exist platforms $l \leq r$ such that condition (6) is satisfied. If instead $r < c$, we have

$$\begin{aligned} \Gamma(l_\Gamma(r), r) &= p(l_\Gamma(r), r)(r - l_\Gamma(r)) = p(l_\Gamma(r), r)^2 \left[p'(l_\Gamma(r), r)\right]^{-1} \\ &= 2\Phi\left(\frac{l_\Gamma(r) + r}{2}\right) \frac{\Phi\left(\frac{l_\Gamma(r)+r}{2}\right)}{\phi\left(\frac{l_\Gamma(r)+r}{2}\right)} > 0 \, , \end{aligned}$$

where I use the first-order condition for a maximum of Γ in l. By $l_\Gamma(r) < r$ and the quasi-concavity of Φ, we further have

$$\Gamma(l_\Gamma(r), r) < 2\Phi\left(r\right) \frac{\Phi\left(c\right)}{\phi\left(c\right)}$$

for any $r < c$. Because $\lim_{r \to -\infty} \Phi(r) = 0$, we can conclude that $\Gamma(l_\Gamma(r), r)$ converges to 0 for $r \to -\infty$. Finally, the derivative of $\Gamma(l_\Gamma(r), r)$ in r is strictly positive, as

$$\frac{d\Gamma(l_\Gamma(r), r)}{dr} = \Gamma_2(l_\Gamma(r), r) = p'(l_\Gamma(r), r)(r - l_\Gamma(r)) + p(l_\Gamma(r), r) > 0 \, .$$

Hence, there is a unique number $r_0 \in (-\infty, c]$ such that $\Gamma(l_\Gamma(r_0), r_0) = c$, and $\Gamma(l_\Gamma(r), r) > c$ for any $r > r_0$.

Third, for any $r > r_0$, the strict quasi-concavity of Γ implies that $\underline{b}(r) \coloneqq \min\{l \in \mathbb{R} : \Gamma(l, r) = c\}$ and $\bar{b}(r) \coloneqq \max\{l \in \mathbb{R} : \Gamma(l, r) = c\}$ are uniquely defined with $\bar{b}(r) < l_\Gamma(r) < b_m(r)$, and that condition (6) is satisfied if and only if $l \in \left[\underline{b}(r), \bar{b}(r)\right]$. □

Proof of Lemma 5

In the following, I prove Lemma 5 through Lemmas 11 to 14.

Lemma 11. *The pair of platforms (l, r) cannot be part of an* entry-stable *allocation if there exists a number $k \in (l, r)$ such that the following two conditions are satisfied:*

$$\Gamma(k, r) > \Gamma(l, r) \text{ , and} \tag{A.1}$$

$$\tilde{Z}(k,l,r) \;\coloneqq\; \Gamma(k,r) + 2p(l,r)\left(k - \frac{l+r}{2}\right) - c > 0\ . \tag{A.2}$$

Proof. Consider an allocation with platforms r and $l < l_\Gamma(r)$. By Lemma 2, platform l can represent the outcome of a primary subgame equilibrium if and only if it is weakly more moderate than the party median, $l \geq m^L$, and if it maximizes $\Gamma(w_i, r)$ over the bliss point of party L's members. If an independent agent with bliss point $k \in (l,r)$ such that inequality (A.1) holds joins party L, he is ensured to win the primary. Entrant k would profit from this entry if the resulting policy gain would exceed the membership cost c, i.e., if

$$-p(k,r)0 - [1-p(k,r)]\,(r-k) - c \;\;\geq\;\; -p(l,r)(k-l) - [1-p(l,r)]\,(r-k)\ ,$$

which can be rearranged to get inequality (A.2). □

Lemma 12. *Define* $\theta(l,r) \coloneqq \max\{k < r : \Gamma(k,r) = \Gamma(l,r)\}$. *For any pair* (l,r) *such that* $\Gamma(l,r) \geq c$, *there exists a number* $k \in (l,r)$ *that satisfies conditions* (A.1) *and* (A.2) *if and only if the inequality*

$$Z(l,r) = \tilde{Z}(\theta(l,r), l, r) = 2p(l,r)\,[\theta(l,r) - l] - c > 0 \tag{A.3}$$

is satisfied.

Proof. The Lemma implies that the entry-preferences of independent agents satisfy the single-crossing condition by Gans & Smart (1996) on the interval $k \in [l, \theta(l,r)]$: If entering party L is profitable for an agent with bliss point k, it is also profitable for any agent with $k' \in (k, \theta(l,r)]$. If it is not profitable for an agent with bliss point $k \in (l, \theta(l,r))$, it is not profitable either for any agent with more extreme bliss point $k' \in [l,k)$. The proof involves three steps.

First, for any $l \geq l_\Gamma$, we get $\theta(l,r) = l$: There is no $k \in (l,r)$ such that (A.1) holds. Correspondingly, $Z(l,r) = -c < 0$.

Second, for any $l < l_\Gamma(r)$, we get $\theta(l,r) \in (l,r)$ and $\Gamma_1(\theta(l,r),r) < 0$. If, more precisely, $\theta(l,r) > (r+l)/2$, then we directly get $Z(l,r) > \Gamma(\theta(l,r),r) - c = \Gamma(l,r) - c$, which is weakly positive in any potential equilibrium. As $\Gamma_1(\theta(l,r),r) < 0$ and function $\tilde{Z}$ is continuous in its first argument, there exists $k \in (l, \theta(l,r))$ such that conditions (A.1) and (A.2) are satisfied.

Finally, if $l < l_\Gamma(r)$ and $\theta(l,r) \leq (r+l)/2$, $\tilde{Z}$ is strictly increasing in k for any $k \in (l, \theta(l,r)]$:

$$\begin{aligned}\tilde{Z}_1(k,l,r) &= p'(k,r)(r-k) - p(k,r) + 2p(l,r) > 2p(l,r) - p(k,r)\\ &\geq 2p(l,r) - p(\theta(l,r),r)\ .\end{aligned}$$

By the definition of $\theta(l,r)$, the last line is equal to

$$p(l,r)\left[2-\frac{r-l}{r-\theta(l,r)}\right]=p(l,r)\frac{r+l-2\theta(l,r)}{r-\theta(l,r)}\geq 0\,.$$

Hence, $Z(l,r)>\tilde{Z}(k,l,r)$ for any $k<\theta(l,r)$. By the continuity of $\tilde{Z}$, there exists a number $k\in(l,\theta(l,r))$ such that both conditions (A.1) and (A.2) are satisfied if and only if inequality (A.3) holds. □

Lemma 13. *There is a number $\tilde{r}\in(r_0,2c)$ such that $Z(\underline{b}(r),r)>0$ if and only if $r>\tilde{r}$.*

Proof. For any $r>r_0$, $\theta(\underline{b}(r),r)=\bar{b}(r)$ and $\Gamma(\underline{b}(r),r)=\Gamma(\bar{b}(r),r)=c$ by construction. Hence, we can rewrite

$$\begin{aligned}Z(\underline{b}(r),r) &= 2p\left(\underline{b}(r),r\right)\left[\bar{b}(r)-\underline{b}(r)\right]-p\left(\underline{b}(r),r\right)\left[r-\underline{b}(r)\right]\\ &= p(\underline{b}(r),r)\left(2\bar{b}(r)-\underline{b}(r)-r\right)\,,\end{aligned}$$

which is strictly positive if and only if $\bar{b}(r)>(\underline{b}(r)+r)/2$. $Z(\underline{b}(r_0),r_0)<0$ because $\bar{b}(r_0)=\underline{b}(r_0)<r_0$. Because $\Gamma(0,2c)=2p(0,2c)c>c$ and $\Gamma(-2c,2c)=2c>c$, we have $\bar{b}(2c)>0$ and $\underline{b}(2c)<-2c$ by the quasi-concavity of Γ. Hence, $Z(\underline{b}(2c),2c)>0$. Hence, $Z(\underline{b}(r),r)$ equals zero for at least one level of $r\in(r_0,2c)$. Finally, the difference $2\bar{b}(r)-\underline{b}(r)-r$ is strictly increasing in r for all $r>r_0$ because

$$\begin{aligned}\frac{d\bar{b}(r)}{dr} &= -\frac{\Gamma_2(\bar{b}(r),r)}{\Gamma_1(\bar{b}(r),r)}=-\frac{p'(\bar{b}(r),r)\left[r-\bar{b}(r)\right]+p(\bar{b}(r),r)}{p'(\bar{b}(r),r)\left[r-\bar{b}(r)\right]-p(\bar{b}(r),r)}>1\,,\text{ and}\\ \frac{d\underline{b}(r)}{dr} &= -\frac{\Gamma_2(\underline{b}(r),r)}{\Gamma_1(\underline{b}(r),r)}=-\frac{p'(\underline{b}(r),r)\left[r-\underline{b}(r)\right]+p(\underline{b}(r),r)}{p'(\underline{b}(r),r)\left[r-\underline{b}(r)\right]-p(\underline{b}(r),r)}<-1\,,\end{aligned}$$

where I use that $\Gamma_1(\underline{b}(r),r)>0$ and $\Gamma_1(\bar{b}(r),r)<0$ for any $r>r_0$, and that $\Gamma_2(l,r)>0$ for any $l<r$. □

Lemma 14. *For any $r>\tilde{r}$, there is a unique number $\tilde{b}(r)\in\left(\bar{b}(r),l_\Gamma(r)\right)$ such that $Z(l,r)>0$ for all $l\in\left[\underline{b}(r),\tilde{b}(r)\right)$ and $Z(l,r)\leq 0$ for all $l\in\left[\tilde{b}(r),\bar{b}(r)\right]$. For any $r\leq\tilde{r}$ and $l\in\left[\underline{b}(r),\bar{b}(r)\right]$, $Z(l,r)\leq 0$.*

Proof. It proves helpful to rewrite function $Z(l,r)$ as

$$Z(l,r)=2\Gamma(l,r)-2p(l,r)(r-\theta)-c=2\Gamma(l,r)\left[1-\frac{p(l,r)}{p(\theta,r)}\right]-c\,;\,,\tag{A.4}$$

where I omit the arguments of $\theta(l,r)$ in the interest of readability. For any $r>\tilde{r}$, $Z(\underline{b}(r),r)>0$ by Lemma 13. For any $l\in\left[l_\Gamma(r),\bar{b}(r)\right)$, we have $\theta(l,r)=l$ by construction, which directly yields $Z(l,r)=-c<0$.

It remains to show that Z is strictly decreasing at any root in l for any $l\in(\underline{b}(r),l_\Gamma(r))$.

The derivative of $Z/2$ with respect to l is given by

$$\begin{aligned}\frac{1}{2}Z_1(l,r) &= \Gamma_1(l,r)\left[1-\frac{p(l,r)}{p(\theta,r)}\right]-\Gamma(l,r)\left[\frac{p_1(l,r)}{p(l,r)}-\frac{p_1(\theta,r)}{p(\theta,r)}\frac{d\theta}{dl}\right]\frac{p(l,r)}{p(\theta,r)}\\ &= \Gamma(l,r)\frac{p_1(l,r)}{p(l,r)}\left[1-2\frac{p(l,r)}{p(\theta,r)}\right]+\Gamma(l,r)\frac{p_1(\theta,r)}{p(\theta,r)}\frac{p(l,r)}{p(\theta,r)}\frac{d\theta}{dl}\\ &\quad -p(l,r)\left[1-\frac{p(l,r)}{p(\theta,r)}\right]\\ &< \Gamma(l,r)\frac{p_1(l,r)}{p(l,r)}\left[1-2\frac{p(l,r)}{p(\theta,r)}\right],\end{aligned}$$

where the inequality follows because $l<\theta(l,r)$ ensures that $p(l,r)<p(\theta,r)$, and because $\theta_1(l,r)=\Gamma_1(l,r)/\Gamma_1(\theta,r)<0$. At any root of Z in l, we have

$$\frac{p(l,r)}{p(\theta,r)}=1-\frac{c}{2\Gamma(l,r)},$$

which is strictly larger than $1/2$ for any $l\in(\underline{b}(r),l_\Gamma(r))$ by the definition of $\underline{b}(r)$. Hence, $Z(l,r)$ is strictly decreasing in l at any root. Because Z is continuous in l, this ensures that, for any $r>\tilde{r}$, there is a unique root $\hat{b}(r)$, which is located in $(\underline{b}(r),l_\Gamma(r))$.

The previous arguments also imply that, for any $r\leq\tilde{r}$, Z has no root in its first argument, i.e., that $Z(l,r)\leq 0$ for all $l\in(\underline{b}(r),l_\Gamma(r)]$. □

Proof of Proposition 1

Proof. I start by proving the *only if* part, i.e., that conditions (i) to (iii) are necessary conditions for the existence of an equilibrium with platforms $(l,r)\in\mathbb{R}^2$. For conditions (i) and (ii), this follows directly from Lemmas 4 and 5. For condition (iii), it follows correspondingly by the symmetry of Φ as imposed by Assumption 1. In particular, the symmetry of Φ implies the right-party's effect on the expected policy is $[1-p(l,r)](r-l)=[1-\Phi(\frac{l+r}{2})](-l-(-r))=\Gamma(-r,-l)$ for any $r>l$. By Lemma 4, $\Gamma(-r,-l)\geq c$ holds if and only if $l\leq -r_0$ and $r\in\left[-b_m(-l),-\bar{b}(-l)\right]$. Similarly, by Lemma 5, a moderate independent profits from from entering party L if $-l>\tilde{r}$ and $-r\in\left[\bar{b}(-l),\tilde{b}(-l)\right)$.

I proceed by proving the *if* part, i.e., that conditions (i) to (iii) are sufficient for the existence of a political equilibrium with platforms (l,r). First, party L has to be efficient by Lemma 3, which is ensured if M^L is such that there are $n\in\mathbb{N}:n\in[C/c,C/c+1)$ members each of whom contributes c. Second, l results as equilibrium platform of party L if l equals $\max\left\{m^L,l_M^(r,M^L)\right\}$ by Lemma 2. This is ensured, e.g., if the median party member has bliss point $m^L=l$ and if each other member $i\in M^L$ has a weakly smaller bliss point $\omega_i<l$. In this case, conditions (i) and (ii) ensures $\Gamma(l,r)\geq c$ so that none of the party members in M^L can profitably leave party L. Finally, if $l\geq\hat{b}(r)$ as required by condition (ii), there is no independent agent who can profitably enter party L and become its party leader by Lemma 5. Again, the symmetry of Φ implies that, if condition (iii) is

met, we can construct a membership set M^R such that r prevails as the equilibrium of party R against l, and that neither any party member can profitably leave party R nor any independent agent can profitably join party R. □

Proof of Proposition 2

Proposition 2 is proven in Lemmas 15 to 18 below.

Lemma 15. *The derivatives of the boundary functions* $\bar{b} : (r_0, \infty) \to (-\infty, r)$, $\underline{b}(r) : (r_0, \infty) \to (-\infty, r)$ *and* $\tilde{b} : [\tilde{r}, \infty) \to \big[\underline{b}(r), \bar{b}(r)\big)$ *satisfy*

$$\bar{b}'(r) > 1 , \tag{A.5}$$

$$\underline{b}'(r) < -1 \text{ , and} \tag{A.6}$$

$$\tilde{b}'(r) \in (-1, 5) . \tag{A.7}$$

Proof. Inequalities (A.5) and (A.6) have already been shown in the proof of Lemma 13.

Consider function $\tilde{b}(r)$, implicitly defined by $Z(\tilde{b}, r) = 2\Gamma(\tilde{b}, r)\left[1 - \frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right] - c = 0$, where $\tilde{\theta}(r) \coloneqq \theta(\tilde{b}(r), r)$. For convenience, I suppress the argument of $\tilde{\theta}$ in the following. Using $\Gamma_1(l, r) = \frac{p'(l,r)}{p(l,r)}\Gamma(l, r) - p(l, r)$, implicit differentiation gives

$$\begin{aligned} \tilde{b}'(r) = -\frac{\Gamma(\tilde{b},r)\frac{p'(\tilde{b},r)}{p(\tilde{b},r)}\left[1-2\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right]+p(\tilde{b},r)\left[1-\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right]+\Gamma(\tilde{b},r)\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}\left[1+\frac{d\tilde{\theta}}{dr}\right]}{\Gamma(\tilde{b},r)\frac{p'(\tilde{b},r)}{p(\tilde{b},r)}\left[1-2\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right]-p(\tilde{b},r)\left[1-\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right]+\Gamma(\tilde{b},r)\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}\frac{d\tilde{\theta}}{dl}} > -1 \\ \Leftrightarrow \qquad 2p(\tilde{b},r)\left[1-\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right] > \Gamma(\tilde{b},r)\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}\left[\frac{d\tilde{\theta}}{dl}-1-\frac{d\tilde{\theta}}{dr}\right] , \end{aligned}$$

where the last inequality follows because $\tilde{\theta}(r) > \tilde{b}(r)$ for any $r > \tilde{r}$ and because $\Gamma_1(\tilde{\theta}, r) < 0$ implies that $\frac{d\tilde{\theta}}{dl} - 1 - \frac{d\tilde{\theta}}{dr} = 2\frac{p(\tilde{\theta},r)-p(\tilde{b},r)}{\Gamma_1(\tilde{\theta},r)} < 0$.

Finally, the inequality $\tilde{b}'(r) < 5$ can be rearranged to get

$$\begin{aligned} 4p(\tilde{b},r)\left[1-\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right]+6\Gamma(\tilde{b},r)\frac{p'(\tilde{b},r)}{p(\tilde{b},r)}\left[2\frac{p(\tilde{\theta},r)}{p(\tilde{b},r)}-1\right]-\Gamma(\tilde{b},r)\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}K(5) = \\ \left\{4p(\tilde{b},r)-\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}\Gamma(\tilde{b},r)K(5)\right\}\left[1-\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)}\right]+ \\ \left\{6\frac{p'(\tilde{b},r)}{p(\tilde{b},r)}-\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}K(5)\right\}\Gamma(\tilde{b},r)\left[2\frac{p(\tilde{\theta},r)}{p(\tilde{b},r)}-1\right] > 0 , \end{aligned}$$

where the inequality follows because $\frac{p(\tilde{b},r)}{p(\tilde{\theta},r)} \in \left(\frac{1}{2}, 1\right)$ for all $r > \tilde{r}$, $\frac{p'(\tilde{b},r)}{p(\tilde{b},r)} > \frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}$ by the log-concavity of Φ, $\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}\Gamma(\tilde{b}, r) < p(\tilde{\theta}, r) \leq 2p(\tilde{b}, r)$ by $\Gamma_1(\tilde{\theta}, r) < 0$ and

$$K(x) \coloneqq 1 + \frac{d\tilde{\theta}}{dr} + x\frac{d\tilde{\theta}}{dl} < 2\frac{p(\tilde{\theta},r)-\frac{p'(\tilde{b},r)}{p(\tilde{b},r)}\Gamma(\tilde{b},r)}{p(\tilde{\theta},r)-\frac{p'(\tilde{\theta},r)}{p(\tilde{\theta},r)}\Gamma(\tilde{b},r)} < 2$$

for all $x \geq 1$. □

Lemma 16. *Function $-\tilde{b}(r)$ has a unique fixed point $\bar{r} > \max\{c, \tilde{r}, 2\ \phi(0)\}$ that is implicitly defined by $\bar{r} + \theta(-\bar{r}, \bar{r}) - c = 0$. For all $r > \bar{r}$, $\tilde{b}(r) > -r$.*

Proof. By Lemma 15, $\tilde{b}'(r) < -1$. Hence, the function $-\tilde{b}(r)$ can have at most one fixed point. If this fixed point exists, it represents the unique root of $\bar{Z}(r) = Z(-r, r) = 2p(-r, r)\,[\theta(-r, r) + r] - c = \theta(-r, r) + r - c = 0$. It remains to show the existence of this root.

First, for any $r > 0$, $r = \Gamma(\theta(-r, r), r) = p(\theta(-r, r), r)\,[r - \theta(-r, r)] < r - \theta(-r, r)$ implies that $\theta(-r, r) < 0$. Hence, we have $\bar{Z}(c) = \theta(-c, c) < 0$. For $r = \tilde{r}$, we have $\tilde{b}(r) = \underline{b}(r)$ and, as shown in the proof of Lemma 13,

$$\tfrac{\bar{b}(\tilde{r}) - \tilde{r}}{\tilde{b}(\tilde{r}) - \tilde{r}} = \tfrac{p(\tilde{b}(\tilde{r}), \tilde{r})}{p(\bar{b}(\tilde{r}), \tilde{r})} = \tfrac{1}{2} \Rightarrow \quad p(\tilde{b}(\tilde{r}), \tilde{r}) = \frac{1}{2} p(\bar{b}(\tilde{r}), \tilde{r}) < \frac{1}{2}\,,$$

which implies that $\tilde{b}(\tilde{r}) < -\tilde{r}$. Hence, $\tilde{r}$ must be to the left of any potential fixed point of $-\tilde{b}$, which ensures that $\bar{Z}(\tilde{r}) < 0$. For $r \leq [2\phi(0)]^{-1}$, moreover, we have $l_\Gamma(r) < -r$ by Lemma 9. This also implies that $\theta(-r, r) = -r$, so that $\bar{Z}(r) = -c < 0$ for any $r \leq [2\phi(0)]^{-1}$.

Second, recall that $\theta(-r, r)$ satisfies $\Gamma(\theta(-r, r), r) = p(\theta(-r, r), r)(r - \theta(-r, r)) = r$. For $r \to \infty$, we have $\theta(-r, r) \to 0$ because $\Gamma(0, r) = \Phi(r/2)r \to r$. Hence, we have $\lim_{r \to \infty} \bar{Z}(r) = \infty > 0$. We can conclude that $\tilde{Z}$ has a unique fixed point $\bar{r} > \max\left\{c, \tilde{r}, [2\phi(0)]^{-1}\right\}$. □

Lemma 17. *There exists a symmetric political equilibrium with $l = -r$ if and only if $r \in [c, \bar{r}]$.*

Proof. First, consider the case $r = c$. The policy effects of both parties are given by $\Gamma(l, r) = \Gamma(-r, -l) = 1/2(r - l) = c$. If $c \neq r_0$, c represents either a fixed point of function $-\bar{b}$ or a fixed point of function $-\underline{b}$. Because $\bar{b}'(r) > 1$ and $\underline{b}'(r) < -1$ by Lemma 15, $-r \in \left[\underline{b}(r), \bar{b}(r)\right]$ if and only if $r \geq c$. (For the knife-edge case $c = r_0$, $\underline{b}$ and $\bar{b}$ are only defined for $r > c$, and both functions have no fixed points. In this case, there are no equilibria with $r < c$ by Lemma 4. For any $r > c$, $\Gamma(-r, r) = r > c$ continues to ensure $r \in \left[\bar{b}(r), b_m(r)\right]$.)

Second, $\tilde{b}(r) > -r$ if and only if $r > \bar{z}$ by Lemma 16. Hence, we have $-r \in B_L(r)$ if and only if $r \in [c, \bar{r}]$. By the symmetry of Φ, we also have $r \in B_R(-r)$ if and only if $r \in [c, \bar{r}]$. Finally, recall that $r \geq c$ ensures $r \geq r_0$ by Lemma 4. By Proposition 1, these conditions are necessary and jointly sufficient for the existence of an equilibrium with platforms r and $l = -r$. □

Lemma 18. *In all political equilibria with platforms $(l, r) \neq (-c, c)$, the platform distance $r - l$ is strictly larger than $2c$.*

Proof. Consider some other platforms $r' \in \mathbb{R}$ and $l' \in \mathbb{R}$ with $r' - l' < 2c$. If $p(l', r') \leq 1/2$, the policy effect of party L is $\Gamma(l', r') = p(l', r')(r - l) < c$. If instead $p(l', r') > 1/2$, the policy effect of party R is $\Gamma(-r', -l') = [1 - p(l', r')](r - l) < c$. Hence, (l', r') cannot be equilibrium platforms. Finally, assume that $r' - l' = 2c$. For any $(l', r') \neq (-c, c)$, this ensures that $p(l', r') \neq 1/2$. Hence, the policy effect of either party L or part R is again below c, implying that (l', r') are no equilibrium platforms. □

Lemma 19. *In all political equilibria, the platform distance $r - l$ is strictly smaller than $\bar{d} \coloneqq 2\bar{r} - \frac{2}{3}\left[\tilde{r} + \tilde{b}(\tilde{r})\right] > 2\bar{r}$.*

Proof. By Lemma 15, $\underline{b}'(r) < -1$ and $\tilde{b}'(r) \in (-1, 5)$. The first property implies that $\tilde{r} - \underline{b}(\tilde{r}) > r - \underline{b}(r)$ for any $r < \tilde{r}$. The second property implies that $r - \tilde{b}(r) < \hat{r} - \hat{l}$ for any $r \in (\tilde{r}, \bar{r})$, where $\hat{r}$ and $\hat{l}$ are defined by

$$\hat{l} = \tilde{b}(\tilde{r}) - (\hat{r} - \tilde{r}) = -\bar{r} - 5(\bar{r} - \hat{r}) \Rightarrow \hat{r} = \bar{r} + \frac{1}{6}\left[\tilde{r} + \tilde{b}(\tilde{r})\right] .$$

Hence, we have $r - \tilde{b}(r) < \hat{d} \coloneqq \hat{r} - \hat{l} = 2\hat{r} - \tilde{r} - \tilde{b}(\tilde{r}) = 2\bar{r} - \frac{2}{3}\left[\tilde{r} + \tilde{b}(\tilde{r})\right]$. Because $\tilde{b}(\tilde{r}) < -\tilde{r}$, $\bar{d} > 2\bar{r}$. Note also that $\tilde{b}'(r) < 5$ ensures that $\bar{d} > \tilde{r} - \tilde{b}(\tilde{r})$. □

Proof of Lemma 6

Proof. I start by showing that $m_+^Q - m^Q \leq 2c$ in every one-party equilibrium. Assume this were not the case and consider some independent citizen i with bliss point $w_i \geq m_+^Q$. If this citizen pays $\alpha_i^Q = c$ and enters party Q, M^Q contains an even number of elements with the two medians m^Q and m_+^Q. Both medians receive an equal number of votes in a pairwise vote, while any other element in M^Q looses at least one pairwise vote. Hence, each median member is chosen as candidate with probability $1/2$. The entrant's policy payoff increases by $1/2(m_+^Q - m^Q) > c$. Entering is hence profitable and the original allocation with $m_+^Q - m^Q > 2c$ was not entry-stable. By similar arguments, $m^Q - m_-^Q \leq 2c$ in every one-party equilibrium.

It remains to show that exit-stability requires that one of the conditions (i) and (ii) is satisfied. For condition (i), no party member can benefit from leaving party Q because this would cause the party's inactivity and, hence, a policy payoff of $-\infty$. For condition (ii), if a party member j with $w_j < m^Q$ leaves party Q, then the members with ideal points m^Q and m_+^Q become the new party medians. Hence, the implemented policy is with equal probability given by each of these ideal points. With $m_+^Q - m^Q = 2c$, j's policy payoff is reduced by c. Hence, he is just indifferent between leaving and staying in Q. Note that in this case, moreover, some independent agents with $w_i > m_+^Q$ are indifferent between staying independent and entering Q. □

Proof of Proposition 3

Proof. First, assume there is an equilibrium with platform $q \in \mathbb{R}$ and $\max\left\{m_+^Q - m^Q, m^Q - m_-^Q\right\} \leq 2c$. Assume further that party Q is efficient and that all independent citizens make zero contributions to both parties. In this case, no member of party L has an incentive to become independent, and no independent agent has an incentive to enter Q, because the achievable gain in policy payoff is dominated by the contribution cost of c.

It remains to check whether an independent citizen i has an incentive to enter the inactive party $-Q$ by contributing $\alpha_i^{-Q} > 0$. This can only have an effect on policy if $\alpha_i^{-Q} \geq C$ so that $-Q$ becomes active. As i is the only member of $-Q$, platform $-q$ must equal w_i. This deviation is individually profitable if and only if the policy effect of w_i in a general election against q is large enough. Without loss of generality, assume that $w_i < q$, so that the policy effect is given by $\Gamma(w_i, q)$.

By Lemma 9, the condition $\Gamma(w_i, q) > C$ is satisfied for some $w_i < q$ if and only if $\Gamma(l_\Gamma(q), q) > C$. This condition is satisfied for any $q > C$ because $\Gamma(l_\Gamma(q), q) \geq \Gamma(-q, q) = q > C$. In contrast, the proof of Lemma 4 shows that there is a number $r_0 < c$ such that $\Gamma(l_\Gamma(r_0), r_0) = c < C$. Moreover, $\Gamma(l_\Gamma(r), r)$ is strictly increasing in r. We can conclude that there is a unique number $r_1 \in (r_0, C]$ such that $\Gamma(l_\Gamma(r_1), r_1) = C$, and $\Gamma(l_\Gamma(q), q) > C$ if and only if $q > r_1$. As a symmetric condition has to hold for independent agents with ideal points $w_i > q$, there exist one-party equilibria with platform q if and only if q is in the interval $[-r_1, r_1]$ and if this interval is non-empty, $r_1 \geq 0$.

It remains to show that $r_1 < 0$ if and only if there is a number $z > 2C$ such that $z\Phi(-z/2) > C$. Fix $q = 0$, and consider the entry incentives for an independent citizen with ideal point $w_i = -z$. If he enters, the resulting policy effect is given by $\Gamma(-z, 0) = z\Phi(-z/2)$. Because $\Phi(x) < 1/2$ for any $x < 0$, we have $\Gamma(-z, 0) < z/2$, which is smaller than C for any $z \leq 2C$. If there exists some number $z > 2C$ such that $z\Phi(-z/2) > C$, then a party with platform $q = 0$ cannot run uncontested. Because $\Gamma(l_\Gamma(q), q)$ is strictly increasing in q, r_1 has to be strictly negative. Otherwise, $\Gamma(l_\Gamma(0), 0) < C$, so there is a one-party equilibrium with platform $q = 0$. By monotonicity, r_1 must be located in $(0, C)$. □

Proof of Proposition 4

Proof. Part (i) follows trivially. For part (ii), recall that $\bar{r}$ is implicitly defined by $\bar{r} + \theta(-\bar{r}, \bar{r}) - c = 0$. Implicit differentiation gives

$$\frac{d\bar{r}}{dc} = -\frac{-1}{1 - \theta_1(-\bar{r}, \bar{r}) + \theta_2(-\bar{r}, \bar{r})} = \frac{\Gamma_1(\theta(-\bar{r}, \bar{r}), \bar{r})}{1 - 2p(\theta(-\bar{r}, \bar{r}), \bar{r})} > 0\,,$$

where the positive sign follows because $\theta(-\bar{r}, \bar{r}) \in (-\bar{r}, \bar{r})$ ensures that $p(\theta(-\bar{r}, \bar{r}), \bar{r}) > 1/2$.

For the limit case $c = 0$, $2c = 0$ again follows trivially. The upper bound in symmetric equilibria is given by $\bar{r}_0 = \max\{r \in \mathbb{R} : r + \theta(-r, r) = 0\}$. It must hence satisfy the

condition $l_\Gamma(\bar{r}_0) = -\bar{r}_0$, which is true if and only if $\bar{r} = [2\phi(0)]^{-1}$ by Lemma 9.

For the general upper bound $\bar{d}$, note first that $c = 0$ implies $\tilde{b}(r) = \min\{l \leq r : 2p(l,r)[\theta(l,r) - l] = 0\} = \min\{l \leq r : \theta(l,r) = l\}$. By the quasi-concavity of Γ and the definition of $\theta(l,r)$, this implies $\tilde{b}(r) = l_\Gamma(r)$, where $\Gamma_1(l_\Gamma(r), r) = p'(l_\Gamma(r), r)(r - l_\Gamma(r)) - p(l_\Gamma(r), r) = 0$. Hence, the derivative of $\tilde{b}(r)$ is given by

$$\tilde{b}'(r) = l'_\Gamma(r) = -\frac{p''(l_\Gamma, r)(r - l_\Gamma)}{p''(l_\Gamma, r)(r - l_\Gamma) - 2p'(l_\Gamma, r)} ,$$

which is located in $(-1, 1)$ for all r because $p'(l_\Gamma, r) > 0$ and, by Assumption 1, $(r - l_\Gamma)p''(l_\Gamma, r)/p'(l_\Gamma, r) < (r - l_\Gamma)p'(l_\Gamma, r)/p(l_\Gamma, r) = 1$ (the latter equality holds by the definition of l_Γ). As in the general case, the property $\tilde{b}'(r) > -1$ ensures that $-\tilde{b}(r)$ has a unique fixed point at $r = \bar{r}$. The property $\tilde{b}'(r) < 1$ ensures that $r - \tilde{b}(r)$ is strictly increasing in r for all $r \leq \tilde{r}$. Hence, the platform distance at the symmetric equilibrium $(l, r) = (-\bar{r}, \bar{r})$ is strictly larger than the distance between any other pair of equilibrium platforms. □

Proof of Proposition 5

Proof. The derivative of the lower bound $2c$ with respect to σ is trivially zero. For the remaining statements, note first that $\frac{dp(l,r)}{d\sigma} \gtrless 0$ for $x \lessgtr 0$ by Assumption 2, which implies that $\frac{d\theta(-r,r)}{d\sigma} = -(r - \theta)\frac{dp(\theta,r)}{d\sigma}/\Gamma_1(\theta, r) < 0$ because $r < \theta(-r, r) < r$ for all $r > 0$ and $\sigma > 0$. Hence, implicit differentiation of $\bar{r}$ with respect to σ gives

$$\frac{d\bar{r}}{d\sigma} = \quad -\frac{d\theta(-r,r)/d\sigma}{1 - \theta_1(-r,r) + \theta_2(r,r)} > 0 ,$$

where the denominator is strictly positive as shown in the proof to Poposition 4.

For the limit results, note first that $\lim_{\sigma\to 0}\theta(-r, r) = 0$ for any $r > 0$ because $\lim_{\sigma\to 0} p(0, r)(r - 0) = r = p(-r, r)(r + r)$. Hence, $\lim_{\sigma\to 0}[\bar{r} + \theta(-r, r)] = \lim_{\sigma\to 0}\bar{r} = c$, which implies that $\bar{r}$ converges to c. Hence, the party platforms in every symmetric two-party equilibrium are given by $(l, r) = (-c, c)$.

For any asymmetric equilibrium with $l < -r$, the policy effect of the leftist party goes to $\lim_{\sigma\to 0} p(l, r)(r - l) = 0$. Hence, $\lim_{\sigma\to 0}\underline{b}(r) = -r$ for all $r > r_0$, i.e., all asymmetric equilibria vanish. We can conclude that the equilibrium platforms are given by $(-c, c)$ with distance $2c$ in every two-party equilibrium.

(For completeness, one can also show that, in the limit, $Z(l, r) < 0$ for any $l < -r$ and any $l = -r > -c$, while $Z(l, r) > 0$ for any $l = -r < -c$. Hence, the intersection of $\underline{b}(r)$ and $\tilde{b}(r)$ converges to $\tilde{r} = c$ with $\tilde{b}(\tilde{r}) = -c$ so that $\lim_{\sigma\to 0} 2\bar{r} - 2/3\left[\tilde{r} + \tilde{b}(\tilde{r})\right] = 2c$.) □

Proof of Proposition 6

Proof. Proposition 6 builds on Proposition 6 above, which establishes the existence of one-party with platforms in some interval $[-r_1, r_1]$ if and only if there is no number $z > 2C$ such that $\Gamma(-z, 0) = z\Phi(-z/2) > C$. In the following, I show that there is a number $\sigma_1 > 0$ such that $r_1 \gtrless 0$ if $\sigma \lessgtr \sigma_1$. First, fix $q = 0$ and some number $z > 2C$. $\Gamma(-z, 0) > 2C\Phi(-z/2)$. By Assumption 2, there is a unique $\sigma_z > 0$ such that $\Gamma(-z, 0) > C$ if and only if $\sigma > \sigma_z$. Because $\Gamma(l_\Gamma(0), 0) \geq \Gamma(-z, 0)$, we must have $r_1 < 0$ for any $\sigma > \sigma_z$.

Second, implicit differentiation of r_1 with respect to σ gives

$$\frac{dr_1}{d\sigma} = -\frac{\frac{d\Phi(l_\Gamma(r_1), r_1)}{d\sigma}}{\Gamma_2(l_\Gamma(r_1), r_1)}.$$

The numerator of this bracket if positive (negative) if and only if $l_\Gamma(r_1) + r_1$ is negative (positive), while the denominator is always strictly positive. For any σ such that $r_1 \leq 0$, we must have $l_\Gamma(r_1) < r_1 \leq 0$. Hence, $\frac{dr_1}{d\sigma} < 0$ for any σ such that $r_1 \leq 0$.

Third, we have $\lim_{\sigma \to 0} r_1 = C/2 > 0$, because $\Gamma(-q + \varepsilon, q) = 2q - \varepsilon$ for any $\varepsilon > 0$.

We can conclude that there is a unique number $\sigma_1 > 0$ such that $r_1 > 0$ if and only if $\sigma < \sigma_1$. In this case, there is a set of one-party equilibria with any platform in the interval $[-r_1, r_1]$. For $\sigma = \sigma_1$, instead, we have $r_1 = 0$, which implies that platform q equals 0 in every one-party equilibrium. □

Proof of Corollary 2

Proof. In the limit case of electoral certainty, $\sigma \to 0$, the party platforms are given by $(l, r) = (-c, c)$ in every two-party-equilibrium by Proposition 5. In any of these equilibria, the median voter is independent and his utility equals $-c$.

As shown in the proof to Proposition 6, there exists a one-party equilibrium with platform q if and only if $q \in [-C/2, C/2]$. In any of these equilibria, the utility of the median voter is given by $-|q|$ if he is independent (and lower if he is a member of the uncontested party). For any q in the intervals $[-C/2, -c)$ and $(c, C/2]$, this utility is smaller than $-c$. □

B Figures

Figure 1: The party formation and candidate selection stages

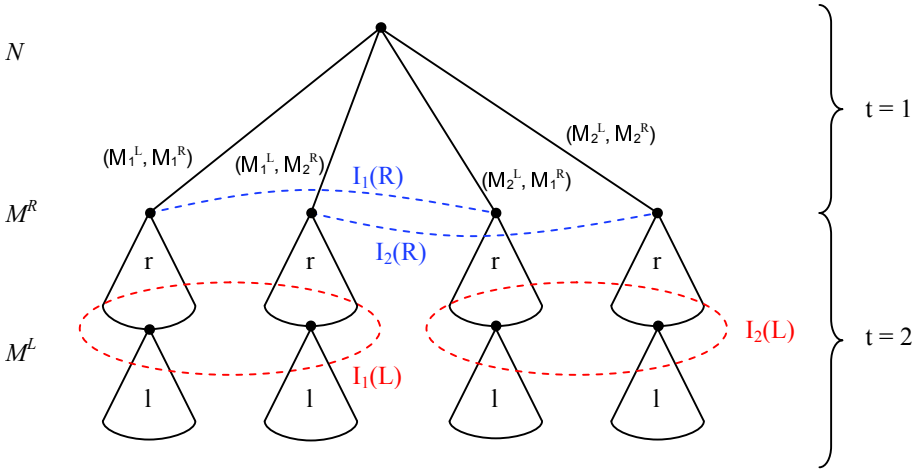
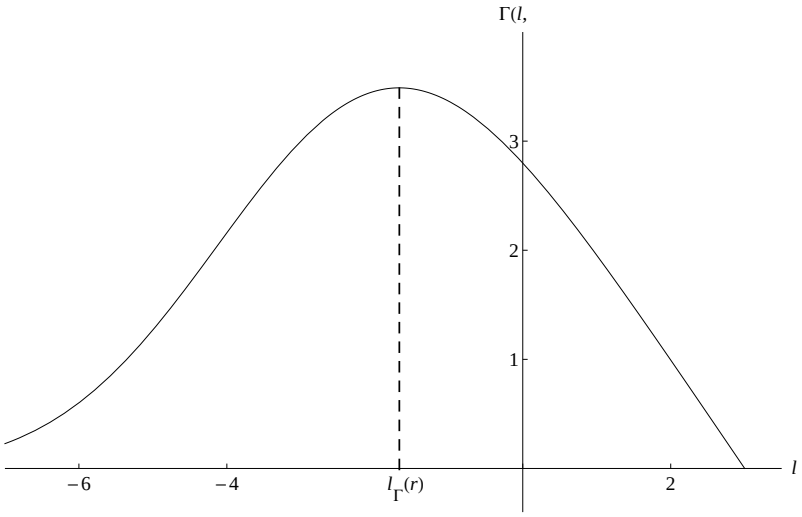
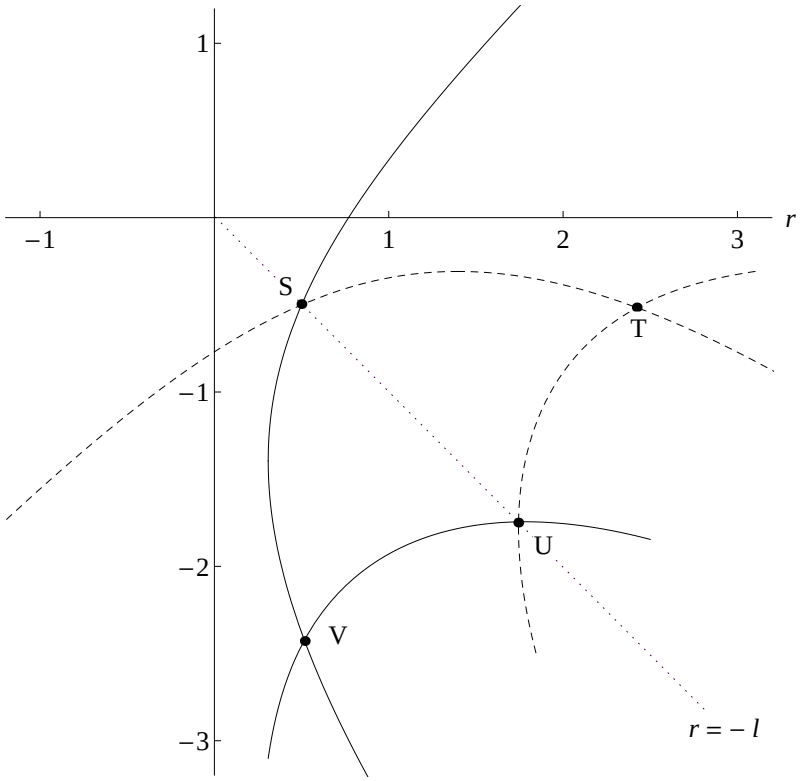


Figure 2: The policy effect function



The figure shows the policy effect function Φ given by a normal distribution with expected value 0 and standard deviation 1. Horizontal axis: Platform l of the leftist party. Vertical axis: Policy effect function $\Gamma(l, r)$ for $r = 3$.

Figure 3: Stable parties and supportable platforms



The figure shows the set $STUV$ of platform combinations (l, r) in political equilibria for $c = .5$ and Φ given by a normal distribution with mean 0 and standard deviation 1. Horizontal axis: Rightist party platform r. Vertical axis: Leftist party patform l.